**Tables and Illustrations of Key Mathematical Properties, Rules, and Number Sets**

**Table 1. Common addition and subtraction situations[[1]](#footnote-1)**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Result Unknown** | **Change Unknown** | **Start Unknown** |
| **Add to** | Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now?2 + 3 = ? | Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two?2 + ? = 5 | Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before?? + 3 = 5 |
| **Take from** | Five apples were on the table. I ate two apples. How many apples are on the table now?5 – 2 = ? | Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat?5 – ? = 3 | Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before?? – 2 = 3 |
|  |  |  |  |
|  | **Total Unknown** | **Addend Unknown** | **Both Addends Unknown[[2]](#footnote-2)** |
| **Put Together/ Take Apart[[3]](#footnote-3)** | Three red apples and two green apples are on the table. How many apples are on the table?3 + 2 = ? | Five apples are on the table. Three are red and the rest are green. How many apples are green?3 + ? = 5, 5 – 3 = ? | Grandma has five flowers. How many can she put in her red vase and how many in her blue vase?5 = 0 + 5, 5 = 5 + 05 = 1 + 4, 5 = 4 + 15 = 2 + 3, 5 = 3 + 2 |
|  |  |  |  |
|  | **Difference Unknown** | **Bigger Unknown** | **Smaller Unknown** |
| **Compare[[4]](#footnote-4)** | (“How many more?” version):Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy? (“How many fewer?” version):Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie?2 + ? = 5, 5 – 2 = ? | (Version with “more”):Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have? (Version with “fewer”):Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have?2 + 3 = ?, 3 + 2 = ? | (Version with “more”):Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have? (Version with “fewer”):Lucy has three fewer apples than Julie. Julie has five apples. How many apples does Lucy have?5 – 3 = ?, ? + 3 = 5 |

**Table 2. Common multiplication and division situations[[5]](#footnote-5)**

|  |  |  |  |
| --- | --- | --- | --- |
|  | **Unknown Product** | **Group Size Unknown**(“How many in each group?” Division) | **Number of Groups Unknown**(“How many groups?” Division) |
|  | **3 × 6 *=* ?** | **3 × ? = 18 and 18 ÷ 3 = ?** | **? × 6 = 18 and 18 ÷ 6 *=* ?** |
| **Equal Groups** | There are three bags with six plums in each bag. How many plums are there in all?*Measurement example*. You need three lengths of string, each six inches long. How much string will you need altogether? | If 18 plums are shared equally into three bags, then how many plums will be in each bag?*Measurement example*. You have 18 inches of string, which you will cut into three equal pieces. How long will each piece of string be?  | If eighteen plums are to be packed six to a bag, then how many bags are needed?*Measurement example*. You have 18 inches of string, which you will cut into pieces that are six inches long. How many pieces of string will you have? |
| **Arrays,[[6]](#footnote-6) Area[[7]](#footnote-7)** | There are three rows of apples with six apples in each row. How many apples are there?*Area example*. What is the area of a 3 cm by 6 cm rectangle? | If 18 apples are arranged into three equal rows, how many apples will be in each row?*Area example*. A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it? | If 18 apples are arranged into equal rows of six apples, how many rows will there be?*Area example*. A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it? |
| **Compare** | A blue hat costs $6. A red hat costs three times as much as the blue hat. How much does the red hat cost?*Measurement example*. A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be three times as long? | A red hat costs $18 and that is three times as much as a blue hat costs. How much does a blue hat cost?*Measurement example*. A rubber band is stretched to be 18 cm long and that is three times as long as it was at first. How long was the rubber band at first? | A red hat costs $18 and a blue hat costs $6. How many times as much does the red hat cost as the blue hat?*Measurement example*. A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first? |
| **General** | *a* × *b* = *?* | *a* × *?* = *p* and *p* ÷*a* = *?* | *?* × *b* = *p* and *p* ÷*b*= *?* |

**Table 3. The Properties of Operations**

Here *a*, *b* and *c* stand for arbitrary numbers in a given number system. The properties of operations apply to the rational number system, the real number system, and the complex number system.

|  |  |
| --- | --- |
| *Associative property of addition* | (*a + b*) *+ c = a +* (*b + c*) |
| *Commutative property of addition* | *a + b* *= b + a* |
| *Additive identity property of 0* | *a +* 0 *=* 0 + *a* = *a* |
| *Existence of additive inverses* | For every *a* there exists –*a* so that *a* + (–*a*) =(–*a*) + *a* = 0. |
| *Associative property of multiplication* | (*a* × *b*) × *c = a* ×(*b* × *c*) |
| *Commutative property of multiplication* | *a* × *b* *= b* × *a* |
| *Multiplicative identity property of 1* | *a* ×1 *=* 1 × *a* = *a* |
| *Existence of multiplicative inverses* | For every *a* ≠ 0 there exists **1**∕***a***so that *a* × **1**∕***a*** = **1**∕***a***× *a* = 1. |
| *Distributive property of multiplication**over addition* | *a* × (*b* + *c*) *= a* × *b* + *a* × *c* |

**Table 4. The Properties of Equality**

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational, real, or complex number systems.

|  |  |
| --- | --- |
| *Reflexive property of equality* | *a* = *a* |
| *Symmetric property of equality* | If *a = b*, then *b = a.* |
| *Transitive property of equality* | If *a = b* and *b = c*, then *a = c.* |
| *Addition property of equality* | If *a = b*, then *a + c = b + c.* |
| *Subtraction property of equality* | If *a = b*, then *a* – *c* = *b* – *c.* |
| *Multiplication property of equality* | If *a = b*, then *a* × *c* = *b* × *c.* |
| *Division property of equality* | If *a = b* and *c ≠* 0, then *a* ÷ *c* = *b* ÷ *c.* |
| *Substitution property of equality* | If *a* = *b*, then *b* may be substituted for *a* in any expression containing *a*. |

**Table 5. Algorithms and the Standard Algorithms: Addition Example**

| **Algorithm** | **Standard Algorithm (for efficiency)** |
| --- | --- |
|   356 +167 400 (Sum of hundreds) 110 (Sum of tens) 13 (Sum of ones) 523 |   11  356 +167 523 |
| ***Note:*** *All algorithms have a finite set of steps, are based on place value and properties of operations, and use single-digit computations.* |

**Table 6. The Properties of Inequality**

Here *a*, *b*, and *c* stand for arbitrary numbers in the rational or real number systems.

|  |
| --- |
| Exactly one of the following is true: *a* < *b*, *a* = *b*, *a* > *b*.If *a* > *b* and *b* > *c* then *a* > *c*.If *a* > *b*, then *b* < *a*.If *a* > *b*, then –*a* < –*b*.If *a* > *b*, then *a* ± *c* > *b* ± *c.*If *a* > *b* and *c* > 0, then *a* × *c* > *b* × *c.*If *a* > *b* and *c* < 0, then *a* × *c* < *b* × *c.*If *a* > *b* and *c* > 0, then *a* ÷ *c* > *b* ÷ *c.*If *a* > *b* and *c* < 0, then *a* ÷ *c* < *b* ÷ *c.* |

**Illustration 1. The Number System**

The Number System is comprised of number sets beginning with the Counting Numbers and culminating in the more complete Complex Numbers. The name of each set is written on the boundary of the set, indicating that each increasing oval encompasses the sets contained within. Note that the Real Number Set is comprised of two parts: Rational Numbers and Irrational Numbers.



1. Adapted from Boxes 2–4 of *Mathematics Learning in Early Childhood*, National Research Council (2009, pp. 32–33). [↑](#footnote-ref-1)
2. These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the = sign does not always mean *makes* or *results in* but always does mean *is the same number as*. [↑](#footnote-ref-2)
3. Either addend can be unknown, so there are three variations of these problem situations. Both Addends Unknown is a productive extension of this basic situation, especially for small numbers less than or equal to 10. [↑](#footnote-ref-3)
4. For the Bigger Unknown or Smaller Unknown situations, one version directs the correct operation (the version using *more* for the bigger unknown and using *less* for the smaller unknown). The other versions are more difficult. [↑](#footnote-ref-4)
5. The first examples in each cell are examples of discrete things. These are easier for students and should be given before the measurement examples. [↑](#footnote-ref-5)
6. The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in three rows and six columns. How many apples are in there? Both forms are valuable. [↑](#footnote-ref-6)
7. Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations. [↑](#footnote-ref-7)