|  |
| --- |
| ESE Star LogoRace To The To LogoFinancial Applications  of Inverse Functions |
| Time is Money. Make Your Money Work for You.  High School (Grade 11-12)  (Updated February 2019) |
|  |
|  |
| This unit is designed to support and fit into a rigorous 4th year mathematics course of higher-level (+) math standards with a focus on financial applications of inverse functions. The unit also integrates the National Standards in Personal Finance Education. Algebra 1 and Algebra 2 are considered prerequisite courses (e.g., using mathematical models, multiple representations of functions, exponential growth, introductory notions of functions and inverse relations). Multiple representations of functions are used to explore the relationships between functions and their inverses; the significance of the meaning of inverse; the need to verify (graphical, tabular, and symbolic) whether inverse relations are, indeed, functions; and the application and interpretation of inverse functions in context. The intent of this unit is to deepen students’ prior learning by introducing and modeling inverse functions in real-world financial situations; financial applications are woven throughout and, in some cases, revisited as students gain new lenses on the mathematical models that support them.  *These Model Curriculum Units are designed to exemplify the expectations outlined in the MA Curriculum Frameworks for English Language Arts/Literacy and Mathematics incorporating the Common Core State Standards, as well as all other MA Curriculum Frameworks. These units include lesson plans, Curriculum Embedded Performance Assessments, and resources. In using these units, it is important to consider the variability of learners in your class and make adaptations as necessary.*  This document was prepared by the Massachusetts Department of Elementary and Secondary Education  Mitchell D. Chester, Ed.D., Commissioner  The Massachusetts Department of Elementary and Secondary Education, an affirmative action employer, is committed to ensuring that all of its programs and facilities are accessible to all members of the public. We do not discriminate on the basis of age color, disability, national origin, race, religion, sex, or sexual orientation.  © 2015 Massachusetts Department of Elementary and Secondary Education (ESE).ESE grants permission to use the material it has created under the terms of the Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License. Additionally, the unit may also contain other third party material used with permission of the copyright holder. Please see Image and Text Credits for specific information regarding third copyrights.  The contents of this Model Curriculum Unit were developed under a grant from the U. S. Department of Education. However, those contents do not necessarily represent the policy of the U. S. Department of Education, and you should not assume endorsement by the Federal Government.  Massachusetts Department of Elementary and Secondary Education, 75 Pleasant St, Malden, MA 02148-4906. Phone 781-338-3300, TTY: N.E.T. Relay 800-439-2370, [www.doe.mass.edu](http://www.doe.mass.edu) |

|  |
| --- |
| Table of Contents  [Stage 1 Desired Results 4](#_Toc373836669)  [Stage 2 - Evidence 5](#_Toc373836670)  [Stage 3 – Learning Plan 7](#_Toc373836672)  [Lesson 1 – Inverse Relationships in Financial Applications 9](#_Toc373836673)  [Lesson 2 – Reasoning about inverse relations in Tables & Graphs 36](#_Toc373836674)  [Lesson 3 – Verifying Inverse Functions Symbolically 58](#_Toc373836675)  [Lesson 4 – Restricting Domains of Non-Invertible Functions 98](#_Toc373836676)  [Lesson 5 – Applications of Logarithmic Functions 114](#_Toc373836678)  [Curriculum Embedded Performance Assessment (CEPA) 132](#_Toc373836681) |

|  |  |  |
| --- | --- | --- |
| Stage 1 Desired Results | | |
| **ESTABLISHED GOALS G**  **National Standards in Personal Finance Ed.**  **FRDM.4** Make financial decisions by systematically considering alternatives and consequences  High-school indicators: Apply systematic decision-making to a long-term goal.  **SI.3** Evaluate investment alternatives.  High-school indicators: Compare the risks and returns of various investments; calculate investment growth given different amounts, times, rates of return, and frequency of compounding; use systematic decision-making to select an investment.  **Mathematics MA Frameworks/Common Core**  **Build a function that models a relationship between two quantities.**  **F-BF.A.1** Write a function (linear, quadratic, exponential, simple rational, radical, logarithmic, and trigonometric) that describes a relationship between two quantities.  **c.** (+) Compose functions.  **Build new functions from existing functions.**  **F-BF.B.4** Find inverse functions algebraically and graphically.  **b.** (+) Verify by composition that one function is the inverse of another.  **c.** (+)Read values of an inverse function from a graph or a table, given that the function has an inverse.  **d.** (+) Produce an invertible function from a non-invertible function by restricting the domain.  **F-BF.B.5** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.  **SMP.2** Reason abstractly and quantitatively.  **SMP.3** Construct viable arguments and critique the reasoning of others.  **SMP.4** Model with mathematics.  **SMP.6** Attend to precision.  **SMP.7** Look for and make use of structure.  **SMP.8** Look for and express regularity in repeated reasoning.  **ELA MA Frameworks/Common Core**  **RCA-ST.CS.4** Determine the meaning of general academic vocabulary as well as symbols, key terms, and other domain-specific words and phrases as they are used in a specific or technical context relevant to grades 11-12 texts and topics. | ***Transfer*** | |
| ***Students will be able to independently use their learning to…***  Apply mathematical knowledge to analyze and model mathematical relationships in the context of a situation in order to make decisions, draw conclusions, and solve problems. **T** | |
| ***Meaning*** | |
| **UNDERSTANDINGS U**  ***Students will understand that…***  **U1** The use of mathematical models to solve financial problems requires careful judgment and consideration of future impact.  **U2** A function may appear to have an inverse in one representation, but must be verified in 3 representations to be certain (tabular, graphical, and symbolic).  **U3** The composition of two functions produces a new function, which can be used to verify whether functions are inverses of each other. | **ESSENTIAL QUESTIONS Q**  **Q1** How can functions and their inverses serve as tools to analyze financial situations?  **Q2** Do all functions have inverses?  **Q3** How does the concept of inverse take on new meaning in the context of exponential and logarithmic functions? |
| ***Acquisition*** | |
| ***Students will know…* K**  **K1** Academic vocabulary in Financial Literacy:  investment, savings, principal, rate, interest rate, yield  **K2** Academic Vocabulary in Mathematics:  inverse, inverse function, inverse notation, symmetry, reflection, 1-to-1, domain, range  verify, compose, composition, invertible, non-invertible function  **K3** The need to verify an inverse function in more than one representation  **K4** Inverse relationship between exponential and logarithmic functions | ***Students will be skilled at…* S**  **S1** Modeling financial situations with functions and their inverses  **S2** Applying inverse functions in the context of a real-world financial problem to evaluate the situation and make decisions  **S3** Using composition to verify the existence of an inverse function from an original function  **S4** Restricting the domain of a function to generate an inverse function, for cases in which an inverse function cannot be verified  **S5** Representing (graphs, tables, symbols, words) and analyzing exponential and logarithmic functions  **S6**  Interpreting and solving real-world problems using functions and inverse relationships  **S7** Defending financial decisions using mathematical evidence  **S8** Using tools (i.e., graphing calculator) strategically and fluently to analyze key characteristics of inverse functions |
| Stage 2 - Evidence | | |
| **Evaluative Criteria** | **Assessment Evidence** | |
| **Content:** Mathematically accurate, supported by credible evidence, multiple representations of functions, verification by composition, evidence of strong reasoning  **Product (Emails & mathematical justification):** Organized, mathematically accurate, appropriate, convincing, logical explanation/ justification, precision (terms, units, symbols)  **Writing:**  Intent captures reasoning, qualitative decision-making, professionalism | **CURRICULUM EMBEDDED PERFORMANCE ASSESSMENT (PERFORMANCE TASKS) PT**  You are a financial planner (Louis Barajas\*). You have three clients with different careers, lifestyles, and expenses (Eddie Romero\*, Connie Tuckman, and Lashonda Jackson). Each client has sent you an email describing their particular concerns, along with a little background about their lifestyles and financial interests. Your goal is to provide the best possible financial plan to address your clients’ financial concerns and help them make sound financial decisions. Your company offers five primary financial investment and savings options; you will evaluate the alternatives and respond to your clients’ emails with recommendations. In your emails, you will provide clear evidence to support your advice, including mathematical models for the different situations, graphical and symbolic representations of the inverse functions that model each client’s situation, and professional communication that provides a mathematical justification for your recommendation. \* Names are referenced from PBS video shown in Lesson 2 Criteria for Success 1. Email Response  Develop an email response for each client (use the templates provided).  Each email (1 for each client) should include:   * A clear and concise description of your recommendation to the client with a proposal for at least one of the five available investment options * A summary of the alternatives you considered, your reason(s) for choosing the options you are proposing, and how you think your recommendation fits with your client’s lifestyle and financial needs * For each option you are recommending, provide at least 1 benefit and 1 risk involved   2. Mathematical Evidence  For each client, develop a collection of evidence that supports your recommendation  Your evidence (a set for each client) should include:   * The mathematical function(s) that model the situation, with an explanation of your reasoning involved in determining the function(s) * Mathematical representation of the investment option(s) that you considered relevant to the situation – include the graphical and symbolic (equation) representations of the functions and their inverses * Verification of the inverse functions that model the problem, using composition * An accurate and logical sequence of calculations and reasoning involved in analyzing and solving the problem for each client’s specific context * A concise, persuasive chain of reasoning connecting your recommendation to the evidence you have provided | |
|  | **OTHER EVIDENCE: OE**  Items marked with asterisks (\*) below may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.  **Lesson 1**   * Pre-Lesson: “Comparing Investments” from MARS (See Instructional Tips) * Pre-Assessment: Functions in Context (Handout 1) * Investigation, Part 1: Opener (Handout 3) \* * Investigation, Part 2: Guided Inquiry, Generalization section (Handout 4) \* * Investigation, Part 3: Further Exploration (Handout 5) (Problem 5) \* * Initial Conclusions about Relations and their Inverses (Handout 6) \* * Math Journal   **Lesson 2**   * Recap: Functions and Inverse Relations \* * Formative Assessment: Conclusions from Graphs (Handout 3) * Graphical Investigation of Inverse Relations (Handouts 4 & 5) \* * Formative Assessment: Visualizing Symmetry (Optional) * Ticket to Leave and Peer Review: Finding Inverse Functions Graphically   **Lesson 3**   * Tabular/Numerical Investigation of Inverse Functions, Part 1 (Handout 1) \* * Applying Generalizations: Relationships between Functions and Inverse Relations (Handout 4) \* * Consumer Spending (Handout 7) \* * Understanding Composition: Frayer Model (last question on the page, “Inference”) (Handout 8) \* * Verifying Inverse Functions Symbolically (Handout 11) \* * Ticket to Leave (formative assessment)   **Lesson 4**   * Finding the Inverse Symbolically (Handout 1) (“Synthesis” reflection, see Lesson 4 Lesson Details) * Exit Ticket (formative assessment)   **Lesson 5**   * Review of Logarithms (Handout 3) (Optional) * Investigating the Inverse of Exponential Functions, Part 2 (Handout 4) \* * Practice Round: Composition with Exponents and Logs * Composing Exponential and Logarithmic Functions (Handout 5) (Problem #3) \* | |
| Stage 3 – Learning Plan | | |
| ***Summary of Key Learning Events and Instruction***    Multiple representations of functions are used to explore the relationships between functions and their inverses, and the significance of the meaning of inverse, in greater depth than prior grades/courses. Financial applications are woven throughout, sometimes revisiting the same problems through new lenses. For example, Financial application problems involving finding *time t* for an investment build from graphically analyzing the situation in lessons 1 and 2, to an understanding of composition and using composition to verify inverse functions in lessons 3 and 4, to finding inverse functions symbolically in lesson 5.  **Lesson 1** This lesson expands on the concept of inverse relations from prior grades/courses to develop the meaning of inverse in the context of inverse functions. Students begin by exploring a $1 million savings problem and then investigate an investment function using graphs and tables to observe the relationships between a function and its inverse. In particular, students will make generalizations about key features such as domain and range, and relate their prior understanding of the definition of a function to an emerging perspective on the meaning of an inverse function.  **Lesson 2** This lesson continues to expand on the meaning of inverse relations and inverse functions using multiple representations, through an in-depth exploration of the graphical representations of functions and their inverses. Students explore the concepts of symmetry and transformation (reflection) when reasoning about functions defined by a graph and/or table, and their corresponding inverse relations. They begin to realize that not all functions have inverses, and that not all inverses of an original function are actually functions. Financial application problems are revisited through graphical analysis and interpreting graphical representations in the context of the real-world situation.  **Lesson 3** This lesson moves from graphical exploration of inverse functions and relations developed in Lessons 1 and 2, to a symbolic investigation of the concept of inverse. Building on the need to verify whether an inverse relation is a function, students are introduced to composition of functions as a means for verifying inverse functions symbolically. Students use a Frayer Model, a visual model, and a simple financial application (consumer spending) to develop a deeper understanding of composition and its application to inverse functions. Students discover a principle of inverse functions as the result of the composition and its inverse, and they express this principle symbolically as (f(f-1(x)) = x and f-1(f(x)) = x.  **Lesson 4** In prior lessons, students have analyzed and verified whether graphical, tabular, and symbolic representations of inverse relations were, indeed, functions and their inverses. In this lesson, students learn to actually *find* the symbolic representation (equation) of an inverse function (given a function that has an inverse, i.e., an invertible function). They discover this by applying the principle of inverse functions discovered in Lesson 3, to solve for an inverse function. This approach develops deep meaning of the concept of inverse, providing a departure from the traditional approach of switching x’s and y’s. The strategy of creating an invertible function from a non-invertible function is also introduced (by restricting the domain of the function), with the goal of solidifying students’ understanding of identifying a unique inverse function.  **Lesson 5** This lesson builds upon students’ prior knowledge of logarithms in Algebra 2 standard F.LE.4 (i.e., as expressions for the solutions to exponential equations/models). Now, students will formalize the meaning of a logarithmic *function* as the inverse of an exponential *function*. Using the principle of inverse functions, f(f-1(x)) = x and f-1(f(x)) = x, students will find the inverses of both exponential functions and logarithmic functions. They will revisit financial applications and apply their learning to solving investment problems symbolically, especially for *time (t)*, whereas previously they were limited to solving graphically with or with a table. | | |

# Lesson 1 – Inverse Relationships in Financial Applications

**Time (minutes):** two 60-minute periods

**Overview of the Lesson**

This lesson expands on the concept of inverse relations from prior grades/courses to developing the meaning of inverse in the context of inverse functions. Students begin by exploring a $1 million savings problem, and then investigate an investment function using graphs and tables to observe the relationships between a function and its inverse. In particular, students will make generalizations about key features such as domain and range, and relate their prior understanding of the definition of a function to an emerging perspective on the meaning of an inverse function.

As you plan, consider the variability of learners in your class and make adaptations as necessary.

**Standard(s)/Unit Goal(s) to be addressed in this lesson:**

**F-BF.B.4.c** (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

**FRDM.B.4** Make financial decisions by systematically considering alternatives and consequences

**SMP.2** Reason abstractly and quantitatively

**SMP.3** Construct viable arguments and critique the reasoning of others

**SMP.4** Model with mathematics

**Essential Question(s) addressed in this lesson:**

How can functions and their inverses serve as tools to analyze financial situations?

**Objectives**

* Describe the concept of inverse through the lens of multiple representations of functions (i.e., tables, graphs, words)
* Formalize the definition of inverse functions using inverse notation

**Language Objectives**

* Relate mathematical terms to financial terms

**Targeted Academic Language**

* Mathematical terms: inverse, inverse notation
* Financial terms: investment, savings, principal, rate, interest rate, yield

**What students should know and be able to do before starting this lesson**

* Basic understanding of the definition of a function as a rule that assigns only 1 output, or element of the range, to each input, or element of the domain (8.F.1, F-IF.1)
* Analyze different functions and their behavior using multiple representations
* Graph a variety of function types to identify and describe key features of their graphs (F-IF.7)
* Identifying domain and range of a function
* Use and interpret function notation f(x) in real-world contexts (F-IF.2)
* Translate between descriptive, algebraic and tabular, and graphical representations of functions
* Recognize how, and why, a quantity changes per unit interval
* Basic understanding of linear growth and exponential growth

**Anticipated Student Pre-conceptions/Misconceptions**

* Students may have difficulty interpreting key features of the graphs or symbolic representations of functions in the context of a real-world problem, even if they are able to find the correct solutions or represent the function graphically.
* Students may confuse the inverse function notation f-1(x) with the reciprocal function notation 1/f(x)

**Instructional Materials/Resources/Tools**

* Graphing calculator or online graphing tool
* Graph paper (optional)
* *Pre-Lesson: “Comparing Investments” from MARS* [*http://map.mathshell.org/materials/lessons.php*](http://map.mathshell.org/materials/lessons.php)*(See Instructional Tips below)*
* “Save and Invest” article: <https://www.dallasfed.org/~/media/documents/cd/wealth/wealth.pdf>

**Instructional Tips/Strategies/Suggestions for Teacher**

Pre-Lesson: “Comparing Investments” MARS Lesson is recommended before beginning Lesson 1 (see Instructional Materials above). The pre-lesson is designed to assess students’ familiarity and flexibility with interpreting exponential and linear functions in the context of financial application (simple and compound interest problems), including how and why a quantity changes per interval and translating between multiple representations of functions.

Additional instructional notes are embedded in the Lesson description below, indicated by the following symbol: >>>

**Assessment**

* Pre-Lesson: “Comparing Investments” from MARS (See Instructional Tips above)
* Pre-Assessment: Functions in Context (Handout 1)
* Investigation, Part 1: Opener (Handout 3) \*
* Investigation, Part 2: Guided Inquiry, Generalization section (Handout 4) \*
* Investigation, Part 3: Further Exploration (Handout 5) (Problem 5) \*
* Initial Conclusions about Relations and their Inverses (Handout 6) \*
* Math Journal

\* Items marked with asterisks (\*) above may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.

**Lesson Details (including but not limited to:)**

**Lesson Opening**

Pre-Assessment: Functions in Context (HANDOUT 1)

This handout shows three functions in algebraic, graphic, and table format. Students will match the different representations and develop a context for each function. Use the think-pair-share strategy to engage students in discussion after they have completed the work individually. (SMP4 Model with mathematics)

Million Dollar Question (HANDOUT 2)

*If you use only 1 dollar bills, how long would it take you to save $1 million, if you start saving now?*

Have students individually record their guesses to this question. Gather a few quick responses to get a sense of students’ guesses. Ask, “Did anyone have a higher response? Lower?” Gauge and share how students reported their guesses (e.g., in terms of years, months, days, etc.).

*>>> Teacher Note*: This question is very broad and open-ended. Any responses are welcome; just gauge the types of initial ideas.

Students pair up and discuss the questions below. Teams record responses on flipchart.

* Compare and discuss your guesses w/your partner.
* What assumptions did you make in order to guess?
* What additional information might you need?
* Design a strategy to start saving for $1 million.

*>>> Teacher Note*: Strategy doesn’t necessarily need to involve saving $1/day; it’s just an opening hook to prompt discussion.

Visit groups to check on progress and to prompt further thinking. Then wrap-up with a whole class discussion, having teams share/present their strategies and any questions/ideas that emerged.

Guiding Questions (while visiting groups and/or during whole class discussion):

* Which methods might be accurate? Which methods might be more realistic? Is there a difference? Why?
* What are alternate methods?
* After hearing other teams’ ideas, would you change your strategy and/or do you have a preference?

*>>> Teacher Note*: Most students will think about this problem linearly (e.g., $10/month) or in installments. Very few may suggest investing (e.g., doubling an initial amount every 5 years). Be on the lookout for these kinds of answers, or pose the question if no one did, pointing out that the underlying mathematics of saving, exploring a variety of financial applications for doing so, will be a central focus of this unit.

*>>> Teacher Note*: Be on the lookout for a student who may think this problem is impossible, that it would take forever. If no one says it, plant the question, is it even possible? This is a key question to lead-in to the rest of the unit, foreshadowing saving with exponential growth applications.

Keep responses on the flipchart posted around the room, if possible, for future reference later in the unit, for students to compare how much they have learned, in comparison to their initial ideas/assumptions.

*This question is about money, personal decision-making, saving, short-term vs. long-term. In this unit, you will be investigating approaches to saving and managing money, methods that are realistic vs. feasible vs. preferred, and exploring the underlying mathematics that will prepare you to make more informed decisions about your personal finances.*

**During the Lesson**

Word Splash

Do a quick brainstorm of key terms to introduce the upcoming investigation. Have students generate as many words as they can about investments (and related financial terms). Post on the wall and revisit throughout the unit.

*>>> Teacher Note*: For reference on financial literacy terms, and especially for ELL students, provide a list of definitions and/or cognates. Make sure students understand the word “investment” and related terms: savings, principal, rate, interest rate, and yield. Use a current events article to generate discussion about topics relevant to them.

Investigation: Part 1, Opener (HANDOUT 3)

Students explore the graph of an investment as a function of time. Working in partners, students read the graph and identify the meaning of various parts of the graph in context (a given point, the y-intercept, etc.). Review the work as a whole class. (SMP4 Model with mathematics)

Discussion Questions

* What type of function is this? How do we know? How do we interpret it in context? How are the units significant?
* Ask about the meaning, in context, of few points additional points (other than P), and also have students choose their own points and have them describe their interpretation.

*>>> Teacher Note*: The purpose of this introductory part of the investigation is to introduce the idea of investing, seeing an investment as a mathematical function, and its graphical representation. Make sure everyone can read the graph appropriately. Some questions might arise about the given point, P, in terms of units (number of years, months, etc.).

Investigation: Part 2, Guided Inquiry (HANDOUT 4)

Students work individually on the next part of the investigation for about 5-10 minutes, and then join a group (of 2 or 3) to compare their solutions.

As students are working, use the questions below to prompt their thinking. Also check for understanding, to make sure they are interpreting the problem correctly and that their interpretation of the problem in context makes sense. (SMP4 Model with mathematics; SMP2 Reason abstractly and quantitatively)

* What are the appropriate units? What are A(t) and t(A) referring to in this problem? (Describe the context in both words *and* mathematical symbols)
* Are there other points we can use to expand on the table of values?
* What observations are you making about the values in the table? About the graph? How would you describe the relationship/comparison in your own words? Does this idea seem familiar?

Choose a few groups to share their results (with a docu-camera or by sharing highlights of their work on a flipchart or board), then invite any groups that had different solutions than the first few. Ask groups to share, specifically, their preliminary conclusions about the relationship between A(t) and t(A). Gather the ideas, and even if they sound similar, capture the nuances in the different explanations. These will be important in formalizing (below).

*Solutions to Interpretations in Context:* The value of the investment depends on the amount of time. The amount of time depends on the value of the investment.

Formalize the initial conclusions that students made from their initial investigation. Draw on students’ explanations (from flipcharts or board), using their words, and then, as a class, construct a sentence that more formally describes the generalization. For example, students may say they just need to “flip the variables in the chart” or “flip x and y” or “the graph is a mirror image.” Encourage students to see if they can state these observations with greater mathematical specificity. For example, “the output of the function is the input of the inverse function” or “every point on the inverse graph is a reflection of the point on the original graph.” Just gather initial conclusions at this point. These formalizations will be solidified after further investigation in both this lesson and the next.

*>>> Teacher Note*: There might be some confusion in interpreting this problem. Be sure that students are working with t vs. A(t) for the first table, and then, for the second table, A vs. t(A). They should recognize that the given graph represents A(t). They will come to realize, in this lesson, that the graph they generate will represent t(A). The goal is to lead students in a discovery to explore inverse functions (t(A) is the inverse function of A(t)), but without immediately using the term or stating this goal just yet. They may also recall that the idea of dependent and independent variables; prompt students to recognize the switch here.

*>>> Teacher Note*: Students may have a sense of inverse from prior learning in Algebra 2 (standard F-BF.4.a), but the emphasis was on determining the input of a function when the output is known by solving an equation of the form f(x) = c for specific values of c and then generalizing that approach to finding a formula that will give the input for a specific output. They may also know the term inverse from prior learning even in earlier grades, in the context of solving for an unknown using inverse operations, also known as “undoing.” Their prior work was predominantly procedural, numerical, and used symbolic representations (e.g., subtraction as inverse of addition, square root as inverse of square). Students should know how to find the inverse of an algebraic equation/expression by applying inverse operations symbolically; this lesson introduces the concept of inverse as a relationship between 2 functions AND the idea that non-invertible functions can have inverses.

*>>> Teacher Note*: The function in this problem has an inverse, but the notion that not all functions have inverses (over the same domain/range as initial function) will be explored more deeply in Lesson 4.

*>>> Teacher Note*: Typically, generalizations are stronger after multiple examples (MP8). Because so many ideas are being introduced and built upon, a semi-formal generalization is provided to wrap up this portion of the investigation. There is no need to solidify these generalizations just yet; simply use this as an opportunity (formative assessment) to gauge the kinds of initial observations and conclusions that students are starting to make on their own.

1. In a table, the domain and range are switched
2. In a graph, switching the domain yields a new graph with a new range
3. Symbolically, the inverse function can be represented with new notation f-1(x).

Note: Conclusion c is only true when the inverse is, itself, a function. Students may not yet realize that not all functions have inverses, and even for most functions that do have inverses, the inverse relations, themselves, may not be functions. These are big ideas that will emerge throughout this unit; just something to keep in mind at this initial stage, but not necessary to go into details now, as students are simply forming initial observations and conclusions.

Investigation, Part 3: Further Exploration (HANDOUT 5)

Students, in groups of 2 or 3, explore their initial conclusions about inverse functions in greater depth, with more examples. They are given 4 functions (3 graphs and 1 table). They compare the graphical and tabular representations of these functions, while they continue to make observations about the effects on shifting domain and range, using the notation, and interpreting the problem in context.

Questions to guide discussion while students are working in groups:

* What do the various representations tell you about the function? About the inverse?
* Which representation(s) are most useful? In what ways do various representations limit the information gained?

Peer Sharing – Pairs/groups share their work with another group. They review each other’s work, ask each other questions for clarification, and compare to their own work. They discuss each other’s feedback, revise any solutions if necessary, making sure to record their reasoning. Together, they formulate a unified generalization, write on flipchart, and share with whole class. Whole class discussion generates a formal conclusion (next).

Use the following questions to guide students’ partner and group work in making sense of and critiquing each others’ reasoning (SMP3 Construct viable arguments and critique the reasoning of others).

* *What is the relationship between the graph of a function and its inverse?*
* *Does every function have an inverse?*
* *Is the inverse relation always a function?*

>>> *Teacher Note:* Students should take time to re-state and describe each others’ reasoning to fully understand it, and then compare it to their own reasoning, which will better prepare them to critique each others’ reasoning (SMP3 Construct viable arguments and critique the reasoning of others). Use this instructional strategy throughout the unit to help students strengthen their proficiency in this Mathematical Practice standard, constructing viable arguments and critiquing other’s reasoning.

Quick Brainstorm

Gather a range of examples of the ways that students have use the term “inverse” in mathematics, from elementary school up to this point (e.g., inverse operations, reciprocals of fractions, inverse variation, etc.). *Ask:* What is the context now? (functions) What is different in this context from prior uses of the term?

>>> *Teacher Note:* Problem 5 (in Handout 5) is intended as an opportunity for students to reflect, formulate initial conclusions, and summarize their work. Parts a – c revisit the same problems that students worked on in Handout 4. Parts d and e take their reasoning further by applying their work to further interpretation of a real-world financial context (SMP2 Reason abstractly and quantitatively; SMP4 Model with mathematics). Use these questions to generate discussion and draw out students’ ideas before you formalize the meaning of inverse functions (see below); it could also be used as a formative assessment.

Formalizing the Definition of Inverse Function

* Recall the first part of definition from the preliminary conclusions above, in terms of input/output values.
* Expand on that part of the definition with graphical and tabular meanings.

Financial Applications

Ask teams to revisit the functions from Investigation, Part 3. Assign groups 1 out of the 4 problems, to distribute the work across the class (e.g., 2 groups work on problem 1, 2 groups work on problem 2, etc.).

Make the connection from their prior learning on exponential functions to growth. Ask a few students for ideas about how they might characterize investments in terms of growth or decay, just qualitatively in words. (SMP4 Model with mathematics)

Pose a new problem about investment growth. Students compare 3 different functions, all on the same coordinate plane, that are growing at different rates. (Choose any functions you wish; they could represent annual compounding, compounding in less than 12 months, and continuous compounding.) Ask students to match the graphs to the 3 options, and provide their thinking. Students could do this problem individually, as a formative assessment. Ask students to raise their hand to indicate their choices; record the number of students who chose each option. Discuss…

Challenge Question: If an investment function doesn’t have an inverse, what might that mean?

*>>> Teacher Note*: Gauge students’ comfort level with making connections between mathematical representations and real-world situations. For students who need additional help, have them visually and literally map back components of the various representations of the functions to the real-world context (e.g., A(t) at t = 4 represents the amount of money accumulated after 4 years).

*>>> Teacher Note*: The “Challenge Question” foreshadows the idea of restricting domains, which will be addressed later in the unit.

**Lesson Closing**

Formative Assessment: Initial Conclusions about Relations and their Inverses (HANDOUT 6)

This is an opportunity to check students’ understanding to this point, about the connections between tabular and graphical representations of functions and inverse relations. They will apply their learning to think backwards, by setting up problems, in words, for a given verbal representation of a function.

Math Journal

How can you be certain that the symbolic representation of an inverse function truly indicates that it represents a function? How do different representations help you investigate this question? How do various representations limit the potential to answer this question? (SMP2 Reason abstractly and quantitatively)

Homework (“Save and Invest” article @ <https://www.dallasfed.org/~/media/documents/cd/wealth/wealth.pdf>

and HANDOUT 7)

Assign the article “Save and Invest” as a reading that will prepare students for a discussion in Lesson 2 about the final CEPA performance assessment. The article can be given all at once and/or divided up and assigned in chunks.

>>> Teacher Note: To aid with comprehension for students who need it, the article may also be accompanied by the Guided Reading questions and/or conducted as a whole class or small group Guided Reading activity (optional, see Handout 7).

**Lesson 1 Handout 1**

**Pre-Assessment: Functions in Context**

1. In this activity you will be looking at several different functions. You have studied these functions in the past. Match the function models below to the appropriate tables of values and graphs.

|  |  |  |
| --- | --- | --- |
| f(x) = .7x | g(x) = x 0.7 | h(x) = .7 x |
| A \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_    graph representing f(x)=.7x | B \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_  graph representing g(x) = x to the 0.7 power | C \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_  graph representing h(x) = .7 to the  x power |
| D \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_   |  |  | | --- | --- | | x | y | | 0.7 | 0.779 | | 0 | 1.000 | | 1 | 0.700 | | 2 | 0.490 | | 2.5 | 0.410 | | E \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_   |  |  | | --- | --- | | x | y | | 0.7 | 0.490 | | 0 | 0.000 | | 1 | 0.700 | | 2 | 1.400 | | 2.5 | 1.750 | | F \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_   |  |  | | --- | --- | | x | y | | 0.7 | 0.779 | | 0 | 0.000 | | 1 | 1.000 | | 2 | 1.625 | | 2.5 | 1.899 | |

2. Functions are indispensable to scientists, economists, and others to describe the relationships between various quantitative data. Create a context for each function above. Include the functions Domain, Range, and Zeros in your description as well as other characteristics of the function.

**Lesson 1 Handout 1 ANSWER KEY**

1. a. g(x) = x0.7

b. f(x) = 0.7x

c. h(x) = 0.7x

d. h(x)

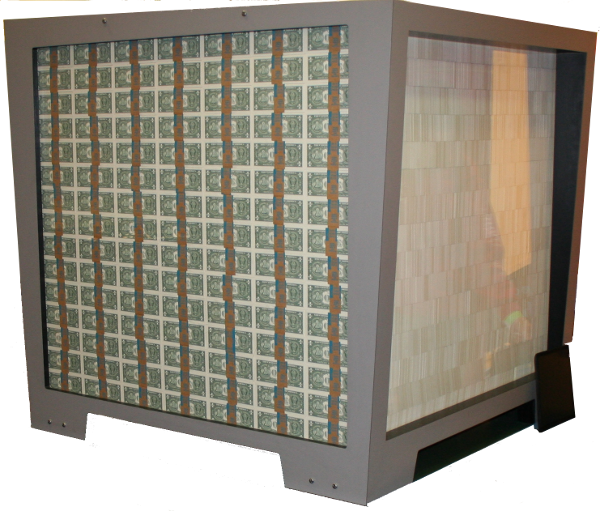
e. f(x)

f. g(x)

2. Answers will vary.

**Lesson 1 Handout 2**

**Million Dollar Question**



The picture above is a million dollars in one dollar bills. If you started saving now, how long do you think it will take to actually have 1,000,000?

1. Take a guess:
2. What assumptions did make in order to make a guess?
3. Design and show one way that this could actually happen.
4. What additional information would you like to have to answer the question?
5. What are some alternate approaches you could take to get to a $1,000,000?

**Lesson 1 Handout 2 ANSWER KEY**

1. Answers will vary.

2. Answers will vary. (Students may have different starting values and growth rates.)

3. Answers will vary.

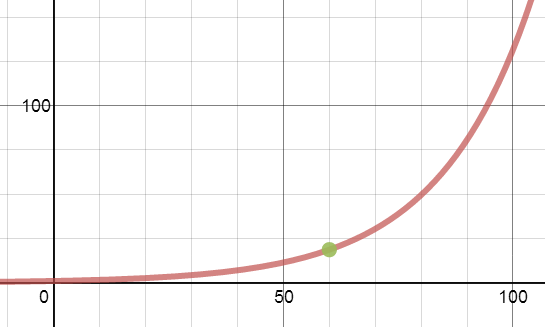
4. Answers will vary. Possible solutions: different starting amount; invest money in stock market; buy and sell a particular product, etc.

5. Answers will vary.

**Lesson 1 Handout 3**

**Investigation, Part 1: Opener**

The graph of the function shows the value of a savings account, *A(t)*, as a function of time, *t*.



P

1. What type of function is shown above? How do you know?
2. What are appropriate units for the *x* and *y*-axis in this context?

*x*-axis:\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

*y*-axis:\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

1. If ***P*** is the point (60, 18.7), what does it mean in the context of the problem?
2. Estimate the value of the y-intercept and describe what it means in the context of the problem.

**Lesson 1 Handout 3 ANSWER KEY**

1. Reasoning about this answer should describe the possibility that it is an exponential function, because the graph appears to approach an asymptote and continues to grow at a more and more rapid rate of change.

2. Answers may vary. Most likely response: x-axis years; y-axis amount (in dollars)

3. After 60 years, the value of the savings account is $18.7 (dollars if y-axis is defined in dollars)

4. (0, 1) The initial value of the savings account was $1.

**Lesson 1 Handout 4**

**Investigation, Part 2: Guided Inquiry**

1. Below is a table of coordinate pairs defined by *A(t)*.

|  |  |
| --- | --- |
| *t* | *A(t)* |
| 0 | 1 |
| 1 | 1.05 |
| 3 | 1.16 |
| 6 | 1.34 |
| 8 | 1.48 |

Use the table of values above to create a **table** (complete the table below) and a **graph** (like the graph you observed in Part 1 (Handout 3) that shows time *t(A)* as a function of the amount in the savings account, *A*.

|  |  |
| --- | --- |
| *t* | *f(t)* |
| 0 | 1 |
| 1 | 1.05 |
| 3 | 1.16 |
| 6 | 1.34 |
| 8 | 1.48 |

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

1. How did you generate the table of values?
2. If you compare the two graphs, what do you notice about them? Write three details that compare the two functions.
3. Let’s take a closer look at the point **P** from the graph of *A(t)* above.
   1. If we look at point **P** (60, 18.7) from above, what would the coordinates be if time were a function of the value of the investment?
   2. Explain what your new point means in the context of the problem.
4. Explain the relationship between *A(t)* and *t(A)* in your own words. Include specific examples from the table and the graph.

**Lesson 1 Handout 4 ANSWER KEY**

5. See table below. Students may use a graphing calculator to make the graph.

|  |  |
| --- | --- |
| **f(t)** | **t** |
| 1 | 0 |
| 1.05 | 1 |
| 1.16 | 3 |
| 1.34 | 6 |
| 1.48 | 8 |

6. Answers will vary. Possible response: I flipped the x- and y-values.

7. Answers will vary. Possible response: symmetrical about the line y = x; they are “flipped” or “mirror images”

8. a. (18.7, 60)

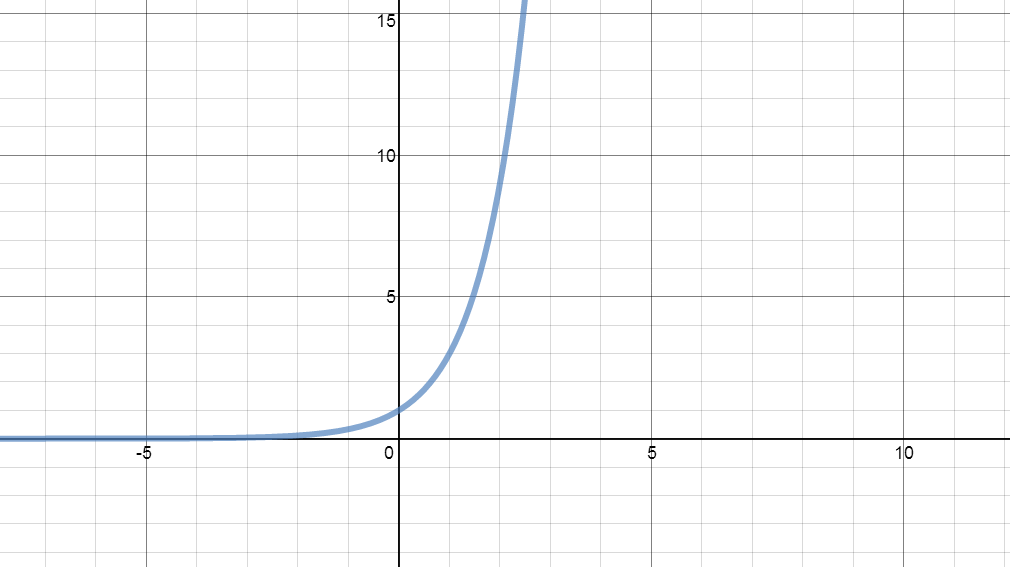
b. If the account has a value of $18.70, then 60 years must have passed (in other words, it took 60 years to save $18.70).

9. Answers will vary. Responses should include the ideas that the values are “flipped” and the graphs are symmetrical.

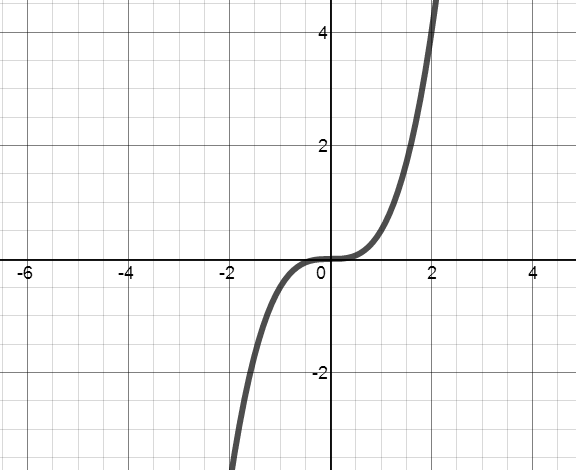
**Lesson 1 Handout 5**

**Investigation, Part 3: Further Exploration**

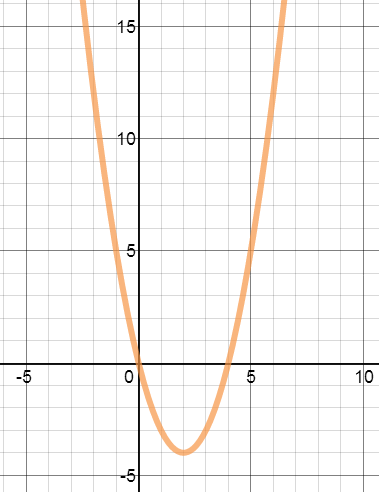
Create tables and graphs of inverse functions from functions provided below.



1. Use the function of h(x) graphed above to answer the specific questions below.
   1. What type of function is this?
   2. Over what interval is the function increasing?
   3. What is the domain and range of the function?
   4. Using the function above create a table of values of *the inverse of the function* and graph *the inverse* on the same graph above*.*
   5. Is *the inverse of the function given above* also a function? How do you know?
   6. What is the domain and range of the inverse of h(x)?
   7. Is *h-1(x)* also a function? How do you know?



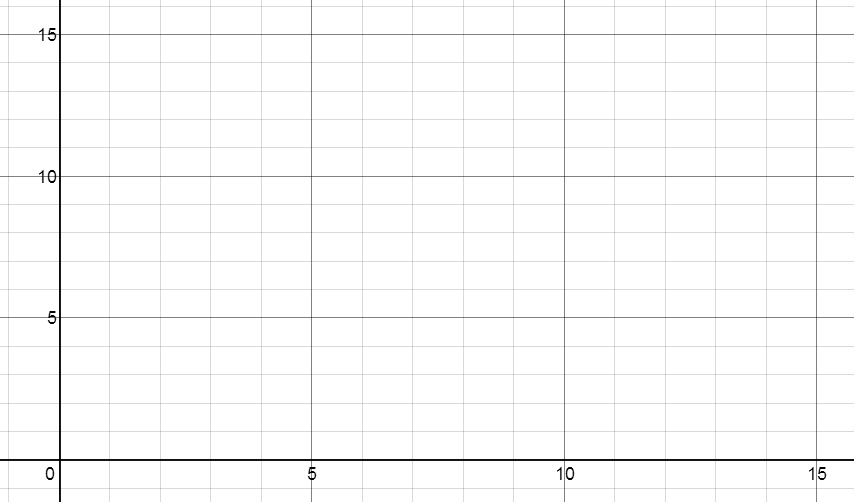
1. Use the function of *f*(x) graphed above to answer the specific questions below.
   1. What type of function is this?
   2. Over what interval is the function increasing?
   3. What is the domain and range of the function?
   4. Using the graph to create a table of values of *f-1(x)* and graph *f-1(x)* on the graph above*.*
   5. Is *f-1(x)* also a function? How do you know?



1. Use the function of *q*(x) graphed above to answer the specific questions below.
   1. What type of function is this?
   2. Over what interval is the function increasing?
   3. What is the domain and range of the function?
   4. Using the graph to create a table of values of *q-1(x)* and graph *q-1(x)* on the graph above*.*
   5. Is *q-1(x)* also a function? How do you know?
2. a. Use the table of *p*(x) below above to create a table of *p-1(x)* .

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | *p(x)* |  | *x* | *p-1(x)* |
| 0 | 0 |  |  |  |
| 2 | 2 |  |  |  |
| 4.5 | 3 |  |  |  |
| 8 | 4 |  |  |  |
| 12.5 | 5 |  |  |  |

b. Graph both *p(x)* and *p-1(x)* on the coordinate plane below.



c. What type of function is *p(x)*?

d. Over what interval is *p(x)* increasing?

e. What is the domain and range of *p(x)*?

f. Is *p-1(x)* also a function? How do you know?

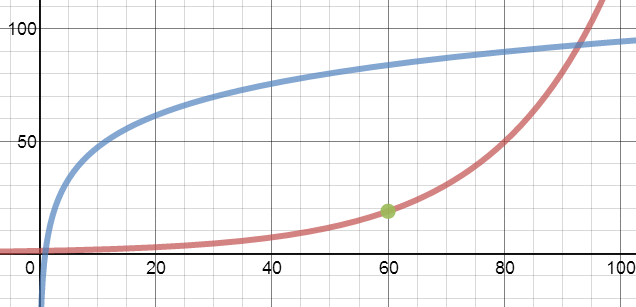
5. Let’s record our findings and start to define the relationship between *A(t)* and *t(A)*.

|  |  |
| --- | --- |
| *t* | *A(t)* |
| 0 | 1 |
| 1 | 1.05 |
| 3 | 1.16 |
| 6 | 1.34 |
| 8 | 1.48 |

|  |  |
| --- | --- |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

How did you generate the table of values for *t(A)*, the time of the investment as a function of the value of the investment?

1. In “our” own words: \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_
2. Using the language of domain and range: \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_
3. What did you notice about the two graphs of the function? The graph of both is shown below:



1. Which graph would you use to find the amount of money you would have after investing for 40 years? Explain.
2. Which graph would you use to find out how long it would take for you investment to triple? Explain.

**Lesson 1 Handout 5 ANSWER KEY**

1. a. Exponential

b. All real numbers

c. Domain: (-∞, ∞); Range: (0, ∞)

d. See table below. (Graph is not shown; use the table to plot points and draw the curve).

|  |  |
| --- | --- |
| **h-1(x)** | |
| **x** | **y** |
| 1 | 0 |
| 3 | 1 |
| 1/2 | -1 |

e. Yes; explanations will vary.

f. Domain: (0, ∞); Range: (-∞. ∞)

g. Yes; for each input there is only 1 output.

2. a. Cubic function

b. Over the set of real numbers (-∞, ∞)

c. Domain and Range: (-∞, ∞)

d. See table below. (Graph is not shown; use the table to plot points and draw the curve).

|  |  |
| --- | --- |
| **x** | **y** |
| 0 | 0 |
| 1 | 2 |
| -1 | -2 |
| 2 | 1.75 |

e. Yes; its graph passes the vertical line test.

3. a. Quadratic function (second degree)

b. Over the interval (-2, ∞)

c. Domain: (-∞, ∞); Range (-4, ∞)

d. See table below. (Graph is not shown; use the table to plot points and draw the curve).

|  |  |
| --- | --- |
| **x** | **y** |
| 0 | 4 and 0 |
| -4 | 2 |
| 5 | 5 and -1 |

e. No; each x input has more than one y output value; the graph fails the vertical line test.

4. a. See table below.

|  |  |
| --- | --- |
| **x** | **p-1(x)** |
| 0 | 0 |
| 2 | 2 |
| 3 | 4.5 |
| 4 | 8 |
| 5 | 12.5 |

b. Graphs are not shown; use the table to plot points and draw the curves.

c. Square Root function

d. Over the interval (0, ∞)

e. Domain and Range: (0, ∞)

f. Yes; each x input has only 1 y output; the graph passes the vertical line test.

5. These are summary questions for students to reflect and make initial conclusions. Refer to Lesson 1 Lesson Details for more description.

a-c. Refer to Lesson 1 Handout 4 for solutions.

d-e. Answers will vary.

**Lesson 1 Handout 6**

**Formative Assessment: Initial Conclusions about Relations and their Inverses**

Create a table, a graph, and an equation that satisfies the following conditions:

An initial Value of 2400 increasing at a rate of 5.3% per year

1. Create a question that can be answered by this function.

2. Create a question that can be answered by the inverse of this function.

**Lesson 1 Handout 6 ANSWER KEY**

Answers will vary.

**Lesson 1 Handout 7 (OPTIONAL)**

**Guided Reading for “Save and Invest” Article**

Refer to the “Save and Invest” article.

1. Summarize what the graph entitled “The Compound Interest Advantage” (page 10) illustrates about investing.

2. Describe at least 3 qualities or attributes of this graph that make it effective at communicating why compound interest does offer an advantage to an investor.

3. Analyze the Investment Pyramid on page 11. What is the relationship between the risk of an investment and the expected return for that investment?

4. Based on the reading, describe what a mutual fund is and how it addresses the risk involved in investing money.

5. Describe 3 big ideas about saving and investing that you learned from this reading.

# Lesson 2 – Reasoning about inverse relations in Tables & Graphs

**Time (minutes):** two 60-minute periods

**Overview of the Lesson**

This lesson continues to expand on the meaning of inverse relations and inverse functions using multiple representations, through an in-depth exploration of the graphical representations of functions and their inverses. Students explore the concepts of symmetry and transformation (reflection) when reasoning about functions defined by a graph or table, and their corresponding inverse relations. They begin to realize that not all functions have inverses, and that not all inverses of an original function are actually functions. Financial application problems are revisited through graphical analysis and interpreting graphical representations in the context of the real-world situation.

As you plan, consider the variability of learners in your class and make adaptations as necessary.

**Standard(s)/Unit Goal(s) to be addressed in this lesson:**

**F-BF.B.4.c** (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

**FRDM.4** Make financial decisions by systematically considering alternatives and consequences

**SMP2**  Reason abstractly and quantitatively

**SMP5** Use appropriate tools strategically

**SMP7** Look for and make use of structure

**Essential Question(s) addressed in this lesson:**

How can functions and their inverses serve as tools to analyze financial situations?

Do all functions have inverses?

**Objectives**

* Make observations and conjectures about inverse relationships graphically, using line of symmetry and reflection
* Generate, compare, and analyze patterns in the values of inverse functions in tables and graphs
* Recognize the need to verify inverse functions, using graphical and tabular approaches

**Language Objectives**

* Use precision in language while observing, describing, and verifying whether functions and relations have inverses (also MP6)
* Begin to gain comfort and familiarity using function notation for inverse functions f-1(x)

**Targeted Academic Language**

symmetry, reflection, 1-to-1, domain, range, function vs. relation

**What students should know and be able to do before starting this lesson**

* Analyze different functions and their behavior using multiple representations
* Describe key features of the graphs of functions (F.IF.7) and understand the relationship between a function’s graph and its domain (F.IF.5)
* Geometric transformations in the coordinate plane, particularly reflection about the x-axis and y-axis (G.CO.2)

**Anticipated Student Pre-conceptions/Misconceptions**

* Students may misinterpret the notation f-1(x) as the reciprocal of f(x) (or 1/f(x)), if they recall the notation of negative exponents on numerical values (e.g., 2-1 = 1/2).
* The specificity in language with which students describe their observations will guide their understanding. Students may associate the terms functions, inverse functions, and inverse relations with incorrect observations.

**Instructional Materials/Resources/Tools**

* Graphing Calculator or online graphing tool
* “Moving Out” Video from PBS Learning Media http://mass.pbslearningmedia.org/resource/fin10.socst.personfin.intro.movingout/moving-out

**Instructional Tips/Strategies/Suggestions for Teacher**

Instructional notes are embedded in the Lesson description below, indicated by the following symbol: >>>

**Assessment**

* Recap: Functions and Inverse Relations \*
* Formative Assessment: Conclusions from Graphs (Handout 3)
* Graphical Investigation of Inverse Relations (Handouts 4 & 5) \*
* Visualizing Symmetry (Optional) \*
* Ticket to Leave and Peer Review: Finding Inverse Functions Graphically

\* Items marked with asterisks (\*) above may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.

**Lesson Details (including but not limited to:**

**Lesson Opening**

Planning for a Solid Financial Future: CEPA Introduction

Preview with students, the CEPA performance assessment task that they will complete at the end of this unit, entitled *Planning for a Solid Financial Future*. Students will consider a variety of investments options as a financial planner and make recommendations to three clients with different lifestyles and financial concerns. Discuss the expectations and criteria for quality. Describe how the next few lessons will prepare students to demonstrate evidence of their learning on the CEPA (refer to Unit Plan Stage 3 for overviews).

Show the PBS “Moving Out” video (see Instructional Materials above) as an opening hook, to engage discussion about saving, managing expenses, planning for post-secondary options, and the importance of financial literacy. The video is also an opportunity to learn more about the role of a financial planner (they will assume the role of Louis Barajas, the financial planner in the video, in the CEPA task, and one of their clients will be Eddie Romero, the college student in the video).

>>> Teacher Note: The CEPA only references characters’ names from the video for the purpose of an authentic and fun connection; however, none of the financial information given in the video is relevant to the CEPA task. *Students should not reference the financial information in the video for the CEPA assignment.*

Review the “Save and Invest” article (assigned in Lesson 1); the article provides critical information that students will need in preparation for the CEPA. Refer back to Lesson 1 for instructional suggestions/options for introducing the article; customize your approach to engaging students in the article based on the variability of your students.

Recap: Functions and Inverse Relations (HANDOUT 1)

In Lesson 1, we made some initial conclusions. Engage students in a recap. Have them partner up and generate as many statements as they can about the observations they made previously, through their exploration of tables and graphs of investment functions, as well as a variety of other functions. They may revisit their reasoning and their partners’ reasoning from the activity in which they formed initial conclusions (from Lesson 1 Handout 4). Then partners should identify the top 3 conclusions that feel most confident about (i.e., those for which they could provide evidence to support). Have pairs switch papers with a different pair, compare their statements, and either confirm or refute the other pair’s statements with an additional piece of evidence. Evidence may include additional reasoning, additional examples of functions, graphical representations, tabular representations, and/or discussion of domain and range.

>>> Teacher Note: Students may not be accurate in all of their reasoning at this point; this is perfectly fine and expected. This is just an initial warm-up to get students comfortable with some ideas, questioning other ideas, and set the stage for deeper investigation. It is also another practice opportunity for the Mathematical Practices of reasoning, justification, and critique (SMP2 Reason abstractly and quantitatively; SMP3 Construct viable arguments and critique the reasoning of others).

Use Handout 1 to provide a more formal recap and summarize this activity.

**During the Lesson**

Observing Graphs (HANDOUT 2)

Using graphing calculators, students (individually) explore the graphs of a variety of functions and their inverses. At this point, they have a nascent understanding of the concept of inverse functions, so this activity is designed as a visual exploration to make observations about the graphical representations. Encourage students to experiment with a variety of different function families, make predictions, and consider why they are getting the results they are seeing. Students should record the equations of the functions they tried, and draw a basic diagram of the shapes of both graphs, original function and its inverse (no grid or values needed).

*>>> Teacher Note*: The use of the graphing calculator tool here is for a nontraditional purpose. A typical goal might be to have students find the inverse of a function by hand and then use the calculator to verify their work, or to find the solution on the calculator and then verify it by hand. By showing students how to simply use the “inverse” function on the calculator, it might appear that the solution is being “given away.” However, the calculator is being used for dynamic exploration through a discovery approach, to make observations rather than to solve, and to analyze qualitatively rather than algebraically (SMP2 Reason abstractly and quantitatively). The fact that students may not yet know how to calculate the inverse equation for themselves is irrelevant here. This exploration will build a foundation for more symbolic manipulation in Lesson 3.

Have students consider the patterns they are observing and form some initial conclusions about the graphical representations of functions and their inverses. Then have students pair up and share three of their explorations with each other, along with a description of their initial observations. Compare to their partner’s graphs and ideas, and test out their own ideas to see if they apply to their partner’s examples. Together, they refine their ideas and make a joint conclusion. Have pairs share their results with the class – they should state both their conclusion and their evidence, based on examples from their explorations. Project examples of the functions that students tried (on graphing calculator overhead, Smartboard, etc., or just draw visually on the board, or have students draw visually on flipchart to share).

Discussion questions to prompt student analysis, both during the exploration and afterward, during whole class discussion:

* What do all the functions that have inverses have in common? What do all the functions that do not have inverses have in common? Are there differences for different function families?
* Are all graphs of functions and their inverse functions symmetrical? Why?
* How can we predict when we are looking at f(x) that the inverse is also a function? Why? What’s the definition of a function?
* How can you tell whether the result of inputting a function into the calculator’s “inverse function” provides the correct result? Why does it matter?
* Did any of your observations seem strange, given what we learned about the domain and range of inverse functions in Lesson 1? Do all functions have inverses? Is the inverse of a function, itself, a function? Always? How can we be sure?

*>>> Teacher Note*: Not all functions have inverse functions. The calculator may not show a graph for every inverse function. This is part of the exploration. Have students consider why this might be. Recall their understanding of switching the domain and range from Lesson 1; the concept of defining an inverse function by restricting the domain will be revisited in Lesson 4. This also engages students in the mathematical practice of using tools strategically (SMP5 Use appropriate tools strategically). Because the calculator will return a graph regardless of whether the inverse function is defined over the given domain, students will need to interpret their results and learn not to take the calculator’s answer at face value.

Pushing students’ thinking in this exploration, especially through the questions that ask whether observations are always true, begs the question of whether there is a more certain way to determine a function that is defined over a given domain (MP2). Students should be noticing symmetry among graphs of an original function and its inverse. Students won’t have all of the answers to these discussion questions; use them simply to whet students’ appetite for and to recognize the need to verify that an inverse function is truly the inverse of a given function, which we will explore in more depth in this lesson. These discussion questions can also be revisited later.

Formative Assessment: Conclusions from Graphs (HANDOUT 3)

To firm up some of the initial conclusions being made about the graphs of functions and their inverse functions, and to check understanding at this point, give students a formative assessment involving visual graphs and verifying which have inverses that are functions, based solely on the graph. Students should be able to justify their reasoning, both visually and with written explanation (SMP2 Reason abstractly and quantitatively). They should also be observing characteristics of the graphs to inform their reasoning (SMP7 Look for and make use of structure).

>>> We will return to this concept later when discuss how we can possibly define the function over a particular domain such that the inverse is also a function.

Student Reflection

Have students partner up and compare their results with each other. To promote reflective and conceptual dialogue (SMP1 Make sense of problems and persevere in solving them; SMP2 Reason abstractly and quantitatively), encourage *students to* *ask each other* these questions:

* Observe any marks your partner made on the graphs on the page. Ask your partner what s/he had in mind.
* Did you feel a need to calculate? Why?
* How confident they are of their responses and what would help them to be more confident.

*Expected Solutions:*

Students should be describing symmetry in terms of the inverse graph as a reflection of the original graph, the need to verify the inverse function, and most importantly, refer back to the definition of a function to discuss the importance of verifying that the inverse function does not have more than 1 output for every input.

Formalization of Inverse Function Concept

Before continuing, take some time to formalize the idea that determining whether the inverse of a function is, in fact, a function depends on the definition of a function (1-to-1 correspondence of inputs to outputs; domain to range). Then formally introduce the horizontal line test. If any students began to recognize, during the opening Observing Graphs exploration, the idea that a graph of an inverse function can be tested visually, or the possibility of a correlation between the inputs and the outputs, draw on their comments, as well.

Our initial conclusions…

* A Function is one-to-one if and only if each second element corresponds to one and only one first element (each x and y value is used only once). Your function will have an inverse if it is one-to-one.
* Graphically we determined whether a function is one-to-one if it passes the horizontal test.
* To apply the horizontal line test, determine if any horizontal line intersects your function intersects you graph only once. If a horizontal line intersects the graph only once the function is one-to-one.
* For a function to be 1-to-1, not only must every element of the domain correspond to only 1 element of the range, but every element of the range must correspond to only 1 element of the domain.

>>> Teacher Note: The last bullet point above may seem obvious, but it will become very important later in the unit as we dive further into the meaning of inverse functions and verifying inverses. For now, let these ideas continue to plant seeds in students’ minds, their understanding will still be emerging.

Problem to try…

Have students test out these conjectures and formalizations by using the graphs shown in the Formative Assessment (Handout 3) to test out the horizontal line test and verify the formal definition of the inverse function.

More discussion prompts:

For each input is there an output? How many? For each output, how many inputs are there? Why is this important to consider?

>>> Teacher Note: The specificity in language with which students describe their observations will guide their understanding. Students may associate the terms functions, inverse functions, and inverse relations with incorrect observations. Be sure to continually encourage students to consider whether a function has an inverse, and if it does, whether the inverse is a function. To help to avoid confusion, students may call an inverse a “relation” until they determine whether it is actually a “function.” The conceptual understanding that is building in this unit is directly tied to the precision with which students describe their observations, make generalizations, and communicate their ideas with language (SMP6 Attend to precision; SMP2 Reason abstractly and quantitatively).

Financial Application of Inverse

Revisit the A(t) investment graph from Lesson 1, and keep students’ recent discoveries handy. Ask students to consider the scenario below (project on board or turn it into a handout). Have students “turn and talk” to a partner to investigate the situation.

*Why is the concept of inverse so integral to financial literacy? Let’s find out.*

*High school seniors typically receive money as graduation gifts. Suppose you collect all of the money you receive for graduation at the end of this year, and bring it to a bank. You open a savings account that yields 2% interest. Model a function that could represent this scenario. If the inverse of the function that models this situation is also a function, what might this tell you about your investment?*

Guidance for teams if they are not sure how to begin:

* The question can be rephrased. *How would you model this scenario with a function A(t)? Graph the function and its inverse. Interpret the graphs in the context of the problem.*
* Consider our emerging conclusions thus far (1-to-1 correspondence between outputs/inputs, graphical symmetry, tabular switching of domain and range, horizontal line test, etc.). Consider the trend being observed in the graph (recall that exponential functions represent growth or decay).

Challenge question: *How do financial planners judge the performance of their investments?*

Gather responses from teams and discuss as a class. Gather as many different responses as possible, paying close attention to students’ words, examples, and graphical evidence for their reasoning (record on board or flipchart). These will be useful later, as students continue to investigate and solidify their notion of inverse functions. (SMP2 Reason abstractly and quantitatively)

*Sample responses:*

Students should be able to recognize, that because of exponential growth, the value of the investment is always increasing. They should also conclude that if the function has no inverse over the given domain, then the investment would increase, but the value would then diminish. The inverse function verifies that there was a turning point.

*>>> Teacher Note*: Some students may not readily make interpret financial implications of the inverse function in context, because not only is the mathematical content new, but the financial applications may also be unfamiliar. Help students map their thinking using a flowchart and reverse flowchart (sometimes called back-mapping). Students consider the input, what is happening to the input both in terms of the behavior/key features of the graph and in terms of the operations on the variable, with each step translating the mathematical representation back to the component of the real-world situation. Then work backwards, mapping out the process on a flowchart in both directions, to uncover the meaning of the inverse in its real-world application.

Graphical Investigation of Inverse Relations, Part 1 (HANDOUT 4)

*The next part of this lesson expands on this analytical approach to studying inverses of functions through their graphical representations. The idea of 1-to-1 correspondence with domain and range should be clearer now. We will now work toward formalizing an expanded definition of inverse functions, to add a graphical meaning.*

Students work in pairs on questions 1 – 6. The goal is to gain fluency in moving back and forth between a function and its inverse function, both graphically and by using the graph to reason about the corresponding values in both functions. These problems, especially #3 and #6, set the stage for later investigation in Lesson 3, in which students will expand their concept of inverse function symbolically. Visit teams to check for understanding and gauge emerging questions. Do not review Part 1 of the investigation as a class yet. Teams will test their conjectures in the next part of the investigation.

*>>> Teacher Note*: Questions 1 – 2 of this investigation involve simply reading the given graph of an exponential function (be sure that students read the problem carefully, to recognize that the given graph is the inverse function f-1(x)). Questions 3 – 5 will require thinking backwards – the ability to visualize the original function and its values f(x) from only the given inverse function f-1(x). Although they should be a bit more comfortable now, based on some similar introductory work in Lesson 1, some students may feel the need to actually see a graph of the original function in order to reason about questions 3 – 5. Scaffolding options include reminding students about their work in Lesson 1 involving transferring outputs of the original function to the inputs of the inverse function, encouraging them to generate a table of values based on the given graph to support their thinking, and if very necessary, giving students a graph of the original function.

>>> *Teacher Note:* Check how students are doing with question 6. They should be making a conjecture about the graphical representations of f(x) and f-1(x), based on their observations/solutions to questions 1 – 5, as well as their earlier graphical/visual observations of qualitative graphs. Especially in question 6, expect students to be more specific now with their generalizations. They should be able to describe their reasoning and justify their conclusions with evidence. (SMP2 Reason abstractly and quantitatively)

Visualizing Symmetry (OPTIONAL)

The next part of the investigation relies on students’ prior knowledge of symmetry and geometric transformations. Before you continue, refresh students’ memories on, particularly, reflections about the x-axis, y-axis, and the line y = x in the coordinate plane. Give students a few problems as a formative assessment to the whole class and/or as extra practice for students who need it. These problems can also be used as additional support later, in Part 2 of the Graphical Investigation (below), for students who may have difficulty recognizing that the symmetry they will be observing is connected to the linear function y = x.

Graphical Investigation of Inverse Functions, Part 2 (HANDOUT 5)

Students will use their graphing calculators, recalling the “inverse function” feature. There are 4 given functions (questions 7-10).

*>>> Teacher Note*: The functions in questions 7 – 10 were chosen because the graphical differences between the given functions and their inverse functions are more visible (SMP7 Look for and make use of structure). Also, these functions cross a variety of function families, giving students a chance to focus on the concept of inverse regardless of the type of function. We will revisit similar (and in some cases, the same) functions later in Lesson 4, via a symbolic approach.

Assign 2 problems to each team (see below). Teams will develop a claim about the graphical relationship between a function and its inverse function. When teams complete their work, they will join another team, switch problems, and test out their claims on the functions they did not try. The group of 4 shares their work, refines or revises their ideas based on their experiences with all 4 problems, and develops a joint conclusion (SMP2 Reason abstractly and quantitatively; SMP3 Construct viable arguments and critique the reasoning of others ). Have groups of 4 record their conjectures, and highlights of their evidence and reasoning, on flipchart (or overhead or paper for docu-camera).

Distributing Teams:

Assign partners to work on 6 different sets of problems. There will be duplicates (e.g., in a class of 24, two sets of partners will be Team A).

Initial Problem-Solving (work on 2 problems) Testing Claims (on remaining 2 problems, then form a group of 4)

Team A – Problems 7 and 8 Team A joins Team F

Team B – Problems 7 and 9 Team B joins Team E

Team C – Problems 7 and 10 Team C joins Team D

Team D – Problems 8 and 9

Team E – Problems 8 and 10

Team F – Problems 9 and 10

Discussion Questions (during the investigation and/or afterward, during whole class discussion):

* What patterns are you noticing? What do the graphical representations of inverse functions have in common, in relation to the original function? What differences do you notice among the different types of functions?
* Does this conjecture apply to all function families? How do we know?
* Why was f(x) = x2 not one of the given functions?
* What is the significance of the line y = x?
* How does the graphical representation help to uncover important relationships between functions and their inverses? How might the graphical representation be limiting?

*>>> Teacher Note*: By now, students should be noticing the significance of the linear function y = x, as a key underlying feature governing the symmetry between graphical representations of functions and their inverses. They may not yet realize why, but recognizing the line should be part of their observations and conclusions. Scaffold for students who need it, by drawing the line y = x on one of their graphs, without telling them which function you’re drawing (they should know). Have them name the function that represents the line you drew, and let them continue discussing on their own, to discover how the line relates to their investigations and emerging conjectures.

*>>> Teacher Note*: Visualizing the symmetry in the pairs of graphs (function and its inverse) about the line y = x is critical here. By this point, students have had multiple experiences with visual, graphical representations of functions and their inverses to form generalizations about the meaning of inverse functions, both in terms of inputs/outputs and visual representation. The significance of the line y = x is a key underlying theme, which will become visible, again, in Lesson 3, when attention shifts to symbolic representations and the definition of inverse is expanded yet again, to include generalizations about symbolic representations.

Conclude the investigation by having representatives from each of the new teams (teams of 4 formed from original pairs) present their findings. Compare and discuss their conclusions. Drawing on students’ responses, summarize/formalize the idea of symmetry about the line y = x, as the graphical basis for the meaning of inverse functions. Refer to discussion prompts, if desired. Foreshadow the next phase of the exploration, the meaning of inverse through symbolic representations (Lesson 3).

Graphical Representations of Personal Finance Functions

Revisit the real-world connection to inverse, by extending the personal finance savings example. Show students the graph of the function A(t) = 1 + 1.08x

*When we found the amount of the investment, we were finding the value of the function A(t). If you have a goal, let’s say buying a car or saving for college expenses, you will need to have an amount in mind that you desire to save. Let’s say you want to save $20,000. The graph shows the value of your savings over time. How much would you have saved after 1 year? After 10 years? How many years would it take to save $20,000?*

*>>> Teacher Note*: The questions that ask how much is saved after a certain amount of time refer to the output of the function, and involve finding the value from the graph. The question that asks how much time is needed to save the value of the investment refers to the inverse of the function, or to find the input from the output. Students can estimate these answers, even without knowing the equation (formula) for compound interest. Instead, they are reasoning about the real-world context, understanding that the values of the investment function and the time based on the investment are inverses of each other, and developing the solution through the graphical representations of the function and its inverse. They will be able to answer the last 2 questions symbolically, later in the unit.

Have students analyze the situation and the context before jumping to solving (SMP2 Reason abstractly and quantitatively; SMP1 Make sense of problems and persevere in solving them). Making sense of the scenario, or interpreting the problem, is more important than the solution here. Have students work on the problem individually, considering the questions below. Then have students “turn and talk,” to share their ideas with a partner. Invite comments from a bunch of students as a whole class, and summarize the main points involved in analyzing this problem.

* Analyze the situation. How is the investment function represented? What values
* How do the first two questions differ from the last question? How would the first two questions be represented graphically, in contrast to the last question?
* Is the graph enough to solve this problem? What other questions could we ask about this scenario? Could we solve those questions with the graph? Why or why not?
* How would you describe the first two questions in terms of input and output? *[Answer: Finding the output in terms of the input, or from the given input]* How would you describe the third question in terms of input and output? *[Answer: Given the output value of the function, find the corresponding input]*

*>>> Teacher Note*: The compound interest formula (below) may or may not be familiar to students, and someone may bring it up. If students were exposed to the formula in prior learning, they may only have evaluated it or solved for a certain value by substituting all of the remaining values. This unit, building on the fourth year (+) standards, provides a deeper investigation of the meaning behind the formula through analysis of multiple representations of inverse functions. Based on their learning up to this point, students are now able to solve for t graphically. There is no need to go into depth here. In Lesson 3, they will extend their learning to symbolic representations of inverse functions, which will enable them to solve for *r* or *n* by reasoning backwards and symbolically (algebraically). In Lesson 4, students will formalize the symbolic representation of inverse functions, which will enable them to solve for *t* symbolically. (Refer back to Lesson 1 MARS Pre-Lesson for additional reference and background).

*>>> Teacher Note*: For students who need additional support making the connection to the scenario, have them underline and circle key words, and map back from the graph and corresponding values on graph to the words of the problem itself.

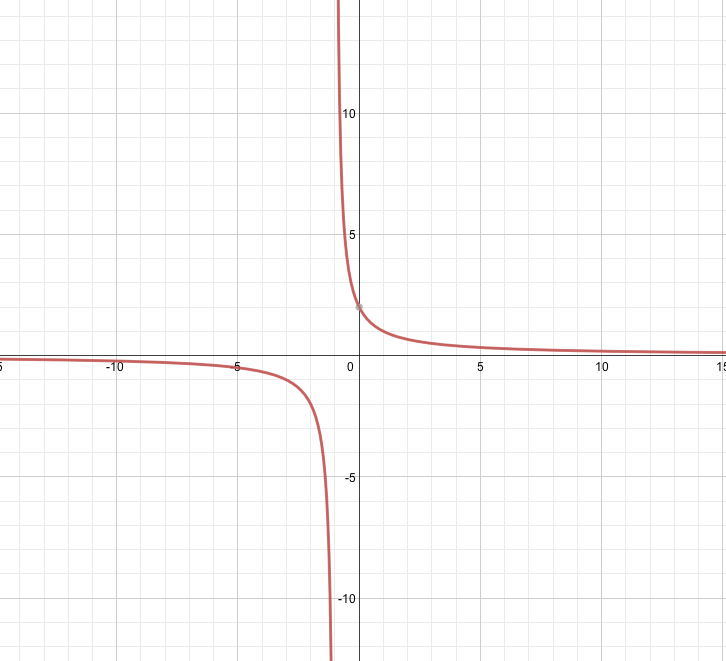
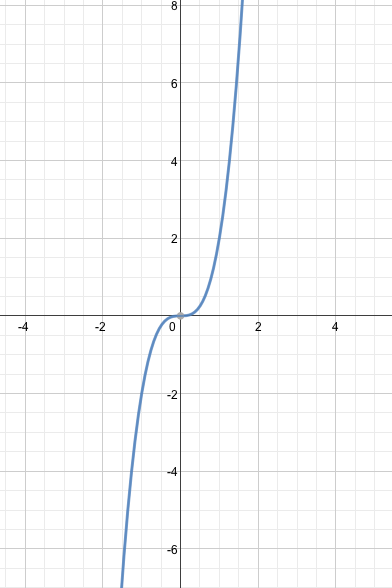
**Lesson Closing**

Ticket to Leave and Peer Review: Finding Inverse Functions Graphically

Give students (on a handout and/or post on the board or overhead) the problems below to gauge and solidify their understanding.

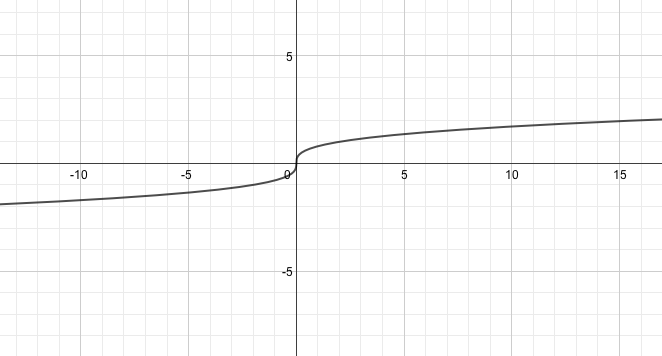
1. Given the graphs of two functions (below), coupled with the line y = x (purely visual, without grid lines), draw the graph of the inverse of each function and justify your solution.

>>> Teacher note: These are the graphs of and f(x) = 2x3

1. The graph below is the inverse of a function (it represents f-1(x)). Explain, in words, how you would determine the graph of the original function, and how you would verify that the graph is correct. Do not make the graph, just describe your process and your reasoning.

>>> Teacher Note: Below is the graph of



Peer Review

Have students pair up and compare their work. Partners ask each other these questions to clarify and justify each other’s reasoning:

1. How did you determine the accuracy of your solution? How did you verify your results?
2. Is your solution true for all values of the function and its inverse? How do you know?
3. How did the line y = x inform your thinking? How did the definition of functions influence your reasoning?

Collect the papers to check for student understanding and inform further instruction. Assign additional examples, for homework or the next lesson, if students need more practice.

Math Journal Reflection

Have students take a few minutes, individually, to consider their learning up to this point.

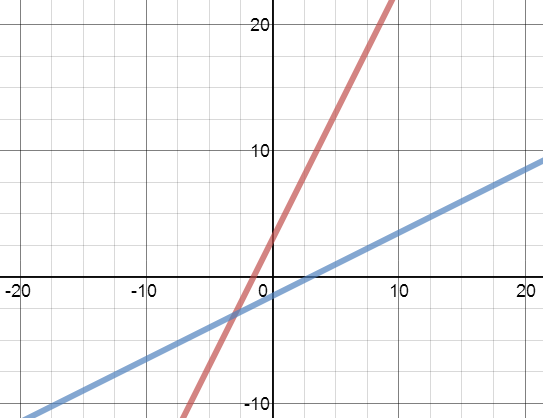
1. What ideas are making sense to you? What was helpful to your learning?
2. What ideas are confusing, or what questions do you still have?
3. Consider and share some preliminary ideas you have about your final project (developing a personal finance plan for post-secondary options). As you think about savings and your personal finances, what might you want to keep in mind based on your learning in this unit so far?

**Lesson 2 Handout 1**

**Recap: Functions and Inverse Relations**

In Lesson 1, we investigated concepts of functions and inverse relations. Our initial conclusions were:

1. A function *f(x)* takes a starting value (an input from the domain) and performs some operation on this value, and yields a value (an output forming a value of the range). The inverse function, *f-1 (x)* takes the output, performs some operation on it and arrives back at the original function’s starting value. Given that these two functions are inverse functions of each other we denote one function *f(x)* and the inverse *f-1 (x)* . Here is a visual representation of that process.



1. Graphically we saw that switching the domain (x-values) with the range (y-values) yielded a new graph . We noted some observations and we will look more at this relationship below.
2. Using tables, we saw that switching the domain (x-values) with the range (y-values) yielded a new table where the domain of f(x) becomes the range of *f-1 (x)* and the range of *f(x)* becomes the domain of *f-1 (x)*.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *x* | *f(x)* |  | *x* | *f-1(x)* |
| -2 | -1 |  | -1 | -2 |
| -1 | 1 |  | 1 | -1 |
| 0 | 3 |  | 3 | 0 |
| 1 | 5 |  | 5 | 1 |
| 2 | 7 |  | 7 | 2 |

**Lesson 2 Handout 2**

**Observing Graphs**

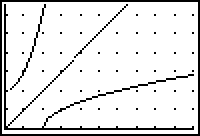
We have already started graphing inverse functions from a given function, today we are going to explore the graphical behavior of inverse functions a bit more deeply. If we know the equation of a function your graphing calculator will draw its inverse.

Directions can be found at : <http://www.tc3.edu/instruct/sbrown/ti83/drawinv.htm>

For this illustration, let’s use **f(x) =**

To draw the inverse of that function:

|  |  |
| --- | --- |
| Paste the Drawing command to your home screen. | [2nd PRGM *makes* DRAW]    *Either* cursor down to the 8 and press [ENTER], *or* simply press [8]. |
| Tell the TI-83/84 to find the original function in Y1. | Press [VARS] [►] [1] [1].  (If your function was in a different numbered y variable, pick that one instead of Y1.) |
| At this point your screen shows this command:      Drawing Y1 |  |
| Now execute the command. | Press [ENTER]. |

The result is shown.

1. Using your calculator, create and graph a variety of functions of your choice. Use the Drawing feature of your calculator to graph the function and its inverse. Use your graphs to determine if the inverses of each of your functions is also a function. Be prepared to share your findings.

**Lesson 2 Handout 2 ANSWER KEY**

1. Answers will vary.

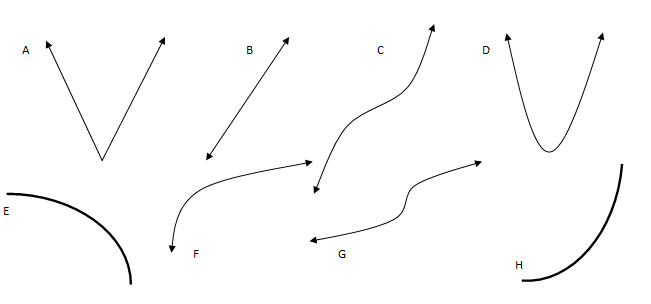
2. Answers will vary. Students may be noticing that graphs with a turning point do not have inverses, or that the graphs of functions that do have inverses are symmetrical.

**Lesson 2 Handout 3**

**Formative Assessment: Conclusions from Graphs**

In the opening investigation, we came to a conclusion that not all functions have inverses that are functions. In other words, the inverse (relation), itself, may not be a function.

Examine the graphs below.



1. Which of these graphs have an inverse that is not a function?

2. How do you know? Justify your reasoning.

**Lesson 2 Handout 3 ANSWER KEY**

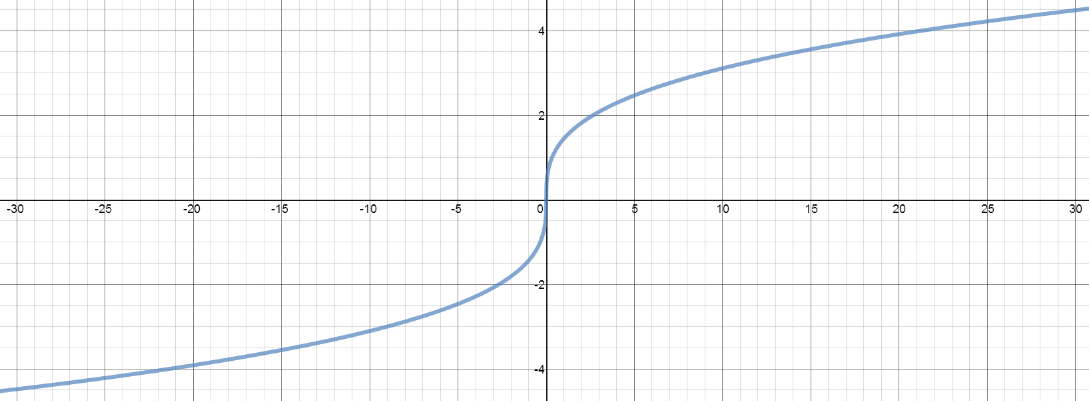
1. A and D

2. Answers will vary. Possible response: The graph does not pass the horizontal line test.

**Lesson 2 Handout 4**

**Graphical Investigation of Inverse Relations, Part 1**

Below is the graph of the **inverse** of a function *f(x)*.



1. Determine the following values based on the graph of the inverse of f(x) shown above.
2. What is the value of f-1 (-3)?
3. What is the value of x, when *f-1 (x)*=-3?
4. Is *f(x)* a function? How do you know?
5. Determine the value of *x* when *f(x)*=3.
6. Determine the value of *f(3)*.
7. Graph *f(x)* on the same coordinate plane, above.
8. Make a conjecture about the relationship between the graphical representations of *f(x)* and *f-1(x).* Explain your reasoning.

**Lesson 2 Handout 4 ANSWER KEY**

1. a. f-1(-3) = -2

b. x = -9

2. Yes. Answers will vary. Possible response: The graph of the inverse passes the horizontal line test.

3. x = 2

4. f(3) = 9

5. (Graph not shown)

6. Answers will vary. Responses should include symmetry of the graphs about the origin or symmetry about the line y = x.

**Lesson 2 Handout 5**

**Graphical Investigation of Inverse Relations, Part 2**

In this exploration, we will test our findings about the graphical relationships between functions and their inverses, and apply them to a variety of functions.

For each function below, using the **Drawing** command on your graphing calculator,

1. Graph the function and its inverse.
2. Describe the graphical relationship between the given function and its inverse.
3. Justify your reasoning.

7.

8.

9.

10.

**Lesson 2 Handout 5 ANSWER KEY**

Students will use their graphing calculators to find and analyze their solutions.

2. What do all the functions that do not have an inverse of a function have in common? How can we predict when we are looking at f(x) that the inverse is also a function?

# Lesson 3 – Verifying Inverse Functions Symbolically

**Time (minutes):** two 60-minute periods

**Overview of the Lesson**

This lesson moves from graphical exploration of inverse functions and relations developed in Lessons 1 and 2, to a symbolic investigation of the concept of inverse. Building on the need to verify whether an inverse relation is a function, students are introduced to composition of functions as a means for verifying inverse functions symbolically. Students use a Frayer Model, a visual model, and a simple financial application (consumer spending) to develop a deeper understanding of composition and its application to inverse functions. Students discover a principle of inverse functions as the result of the composition and its inverse, and they express this principle symbolically as (f(f-1(x)) = x and f-1(f(x)) = x.

As you plan, consider the variability of learners in your class and make adaptations as necessary.

**Standard(s)/Unit Goal(s) to be addressed in this lesson:**

**F-BF.A.1.c** (+)Compose functions.

**F-BF.B.4.b** (+) Verify by composition that one function is the inverse of another.

**SI.3** Evaluate investment alternatives.

**SMP.2** Reason abstractly and quantitatively.

**SMP.3** Construct viable arguments and critique the reasoning of others.

**SMP.6** Attend to precision. \*

**SMP.7** Look for and make use of structure.

**SMP.8** Look for and express regularity in repeated reasoning. \*

\* Although these standards are not listed in the Unit Plan, they are added here because they are particularly relevant to this lesson, in conjunction with the other Mathematical Practices listed.

**Essential Question(s) addressed in this lesson:**

Do all functions have inverses?

**Objectives**

* Compose functions
* Use composition to verify inverse functions symbolically
* Make connections between symbolic representations (equations), graphical/tabular representations, and visual models for inverse functions
* Gain fluency in recognizing the differences between functions that have or do not have inverses – graphical, tabular, and symbolic representations

**Language Objectives**

* Make sense of visual/pictorial representations to expand understanding of mathematical language and concepts about relationships between functions and their inverses
* Gain fluency using and interpreting function notation for inverse functions f-1(x)

**Targeted Academic Language**

Mathematical (Tier 3) terms: compose, composition

Related Tier 2 terms: verify, represent, symbolic

**What students should know and be able to do before starting this lesson**

Writing equations in two or more variables to represent relationships between quantities (A-CED.2)

Representing constraints for equations in two or more variables, and interpreting the reasonableness of solutions in a modeling context (real-world situation) (A-CED.3)

Finding inverses of simple functions by solving an equation, such as the solution to f(x) = 2x3, for different families of functions with non-variable exponents (F-BF.4a)

**Anticipated Student Pre-conceptions/Misconceptions**

* Some students may misinterpret the notation f-1(x) as 1/f(x).
* Students may have difficulty keeping track of which functions and values correspond to the original vs. the inverse function, especially during the Generalization activity (Handout 3)
* The notation for composition f(g(x)) is often misinterpreted as multiplication, because students have become accustomed to using parentheses as an operation. In this case, the operation is the composition of the function, not multiplication. Use the notation f o g as an alternative.

**Instructional Materials/Resources/Tools**

* Graphing calculator or online graphing tool
* Colored pencils/markers (optional)

**Instructional Tips/Strategies/Suggestions for Teacher**

Students come into this unit with a basic definition of function, from prior grades/courses (8.F.1, F-IF.1), as a 1-to-1 correlation between outputs and inputs, or individual values in the domain and range. Through the investigations in Lessons 1-4, we are deepening and expanding on that definition. New ideas include the significance of the domain as it actually becomes the range of the inverse function, and the need to verify that the observed or calculated inverse of a function is, in fact, itself, a function. The significance of inverse functions in financial literacy will continue to be applied throughout this unit.

We will now build a bridge from verification of inverse functions graphically (Lesson 2), to verifying inverse functions symbolically (using composition) here in Lesson 3, which will then lead us into not just analyzing and verifying, but actually finding the symbolic representations (equations) of inverse functions (Lesson 4).

Additional instructional notes are embedded in the Lesson description below, indicated by the following symbol: >>>

**Assessment**

* Tabular/Numerical Investigation of Inverse Functions, Part 1 (Handout 1) \*
* Applying Generalizations: Relationships between Functions and Inverse Relations (Handout 4) \*
* Consumer Spending (Handout 7) \*
* Understanding Composition: Frayer Model (last question on the page, “Inference”) (Handout 8) \*
* Verifying Inverse Functions Symbolically (Handout 11) \*
* Ticket to Leave (formative assessment)

\* Items marked with asterisks (\*) above may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.

**Lesson Details (including but not limited to:)**

**Lesson Opening**

*In the prior lesson, we explored the graphical representations of functions and their inverses to discover some important points. What generalizations did we form from our explorations?*

* *We noticed, graphically, that two functions are inverses of each other if they are symmetrical about the line y = x.*
* *We can determine, graphically and with tables, whether a function is the inverse of an original function by comparing points on both graphs and noting whether the domain of a function becomes the range of the other and if the range of the function becomes the domain of the other.*

Why Are Inverses Important?

This opening discussion is designed to engage students in discussion about the significance of inverse through common everyday examples. Begin with a description/discussion of a cell phone call. When a friend or family member calls you on your cell phone, the caller ID shows the number of the person who is calling, which enables you to recognize the number. You will always recognize the number (and the phone itself will recognize the number through its memory in the contacts list), because each person’s cell phone has a unique number that identifies it. Likewise, your friend’s or family member’s phone will recognize your number. This represents a unique, 1-to-1 correspondence relationship.

Have students work in groups of 3 to explore a few other examples, and encourage groups to generate their own examples. Here are a few to get teams started: ATM machines, Gym memberships, EZ-Pass, GPS navigation.

As they explore these examples, have groups discuss the questions below. Wrap up with sharing out and whole class discussion; highlight emerging themes about inverse functions for students keep in mind during this lesson (see Teacher Note below).

* What function concepts are necessary to make these relationships work?
* What might occur if these relationships were not 1-to-1?
* Why is the correspondence from input to output just as important as the correspondence from output to input (i.e., domain to range vs. range to domain)?
* How does this forward and reverse perspective on the 1-to-1 correspondence between domain and range relate to inverse functions?

>>> Teacher Note: The ideas and generalizations that emerge from this discussion will become very relevant in this lesson, as we explore composition of functions and symbolic verification of inverse functions. Remind students of these ideas, especially struggling students, during the visualization activities involving apples/lemons, as well as other parts of the lesson.

**During the Lesson**

*Today we will expand on our emerging definition of the concept of inverse, this time using tabular/numerical representations of functions and their inverses.*

Tabular/Numerical Investigation of Inverse Functions, Part 1 (HANDOUT 1)

Students, working in pairs, investigate the connections between graphical and tabular (numerical) representations of functions and their inverses. They begin by revisiting the Million Dollar problem from Lesson 1, this time with a graph that models the situation. They use their learning from Lesson 2 to graph the inverse of the given function (reflection about y = x) and interpret the problem in context. Check in with teams to make sure they are starting correctly.

*>>> Teacher Note*: This investigation begins graphically, making the connection to the prior lesson. Then students move into further analysis using tabular representations, and then conclude with symbolic representations. They will explore functions that have an inverse, that have an inverse that is not a function, and that do not have inverses at all. Each example provides a different perspective, which, once analyzed collectively, will open up new understandings. (SMP2 Reason abstractly and quantitatively)

*>>> Teacher Note*: In problem3, students may consider doubling as a linear function. This is typical in their initial explorations; they will gain greater depth of understanding later. Students might use values that do not overlap. Guide them to notice that they need some overlapping values in order to make comparisons (see discussion prompts).

Tabular/Numerical Investigation of Inverse Functions, Part 2 (HANDOUT 2)

Students continue their exploration, being mindful of the key characteristics of the graphs that highlight important information about the functions and their inverses – especially domain and range. Tables will be limited to those that are inverse functions.

Use the prompts below to push students’ thinking, both during the investigation and afterward, during whole class discussion:

* Problem 6. How does the process of determining inverse values in tables compare to the process using graphs? What did you find out from the graphical explorations that could help you here?
* Problem 7. What does part c actually mean? (the inverse of a function is not a number, it is a relation or function, itself)

*>>> Teacher Note*: The notation for composition f(g(x)) is often misinterpreted as multiplication, because students have become accustomed to using parentheses as an operation. In this case, the operation is the composition of the function, not multiplication. Use the notation f o g as an alternative, if it helps, and also to help students begin to use both types of notation interchangeably.

>>>*Teacher Note*: Students may be somewhat frustrated by the tediousness of verifying the inverse of functions using tabular/numerical representations. This is intentional. The goal is to generate a thirst for a more efficient method for verification and to motivate the desire to find inverse functions symbolically (algebraically), while continuing to sharpening students’ ability to reason and think deeply about the meaning of inverse through a discovery approach. All of this is leading to the use of composition to verify inverse functions, a nontraditional approach to studying the concept of inverse, which will be introduced later in this lesson.

Conclude the investigation by having groups share highlights of their work with the class; have students write key ideas on flipchart. The ideas they record and share with the entire class do not need to be a step-by-step recap of their solutions; instead, have groups share their salient observations, interesting patterns they noticed, sections they found puzzling and why, connections they made between this investigation and their prior learning in this unit, etc. Summarize important points emerging from this investigation, using students’ own words and then, as needed, with more formal explanations and notation. (SMP7 Look for and make use of structure; SMP8 Look for and express regularity in reasoning)

Generalizations: Relationships between Functions and Inverse Relations (HANDOUT 3)

Engage students in this investigation of a pictorial representation of inverse functions, to help them think flexibly about the underlying concepts and deepen their understanding by application to a different context. Students should work individually first, and then compare their results with a partner.

>>> Teacher Note: When students are analyzing the apples/lemons image, the purpose is to recognize they have to read the diagram backwards. Some students may do better with re-drawing the diagram in reverse, but with forward arrows. Encourage them to persevere and make sense of the diagram in their own way (MP1); other strategies include breaking it into parts, verbalizing their interpretation, etc. Along these lines, it may be confusing for students to reason that f-1(f(apples) = apples is true. Remind them that the two functions are inverses of each other, so that either function can be called f-1 if we call the other one the function f. For example, we can say f(x) = 2x and f-1 (x) = ½ x AND it is also correct to say f(x) = ½ x and f-1 (x)= 2x. I n other words f(x) and f-1 (x) are notations for functions and their inverses. Students may not readily see this at first, which is expected, but the goal is for them to come to this realization.

>>> Teacher note: This pictorial representation of inverse functions is revisited a few times during the unit. The picture may be very helpful for students who are visual learners, students who have difficulty seeing the concepts, or those who may be seeing the concepts emerging in their heads but are having difficulty expressing it with precision of language (MP6) (note: this third group of learners may or may not be ELLs). Welcome descriptions in students’ own words, and guide them to progress in their sophistication of communicating their ideas mathematically while preserving their conceptual ideas even if their description is not yet so precise. Also encourage students to express their answers pictorially in addition to words, to aid their thinking. This representation could also be explored with objects (e.g., real apples and lemons, or other objects, inside shoe boxes) for tactile learners. On the other hand, this pictorial representation may be more difficult for analytical or logical thinkers to grasp; give those students the option of testing their ideas with numbers to help them reason through it, before they return to the perspective of pictorial representations. Some students may be ready to generalize more quickly, even before they have completed all of the problems; this is perfectly fine; challenge those learners to provide strong evidence through reasoning, examples, and counterexamples (SMP8 Look for and express regularity in reasoning).

Guiding questions to prompt student thinking (for all students):

* f(x) is 1-to-1 function: Ask students to recall what that means. It means the inverse of the function f-1 exists.
* Problem 4a (as an example): f(f-1(apples)) means what? Engage students in discussion, connecting visual representation to verbal explanation to symbolic representations (SMP1 Make sense of problems and persevere in solving them; SMP2 Reason abstractly and quantitatively)
* Can we find f-1(apples)? Students may say that apples are not in the domain of f-1. Use vocabulary of domain and range to encourage reasoning and sense-making (SMP1 Make sense of problems and persevere in solving them; SMP2 Reason abstractly and quantitatively) through developing an emphasis on being precise in the ways they are communicating, verbally, pictorially, and in writing (SMP6 Attend to precision)
* Is the domain different for a **composite** function vs. the **original** function? Why? What does this look like, pictorially and with values?

>>> Teacher note: Composites have not been introduced yet. Have students make some statements about inputs and outputs for f and f-1. For example, “The output of f-1 is the input of f.” Because the inverse relationship is one-to-one, remind them that the input output statement is reversible, so that, “The input of f is the output of f-1” (recall the opening activity). This will help to make the exploration more logical. Discuss that the ‘inner’ function is evaluated first and the output is then evaluated using the ‘outer function’. The end result – which statements are true – will be solidified with composites of inverse functions later in the unit.

Students should be able to think flexibly. After they have completed the problems, ask a challenge question:

* How would our solutions change if we called the first box f-1 and the second box f? How would that change your reasoning (or would it)? Make predictions…

>>> Teacher note: An important point to remember and to lead students toward, is that it doesn’t matter which one we name f, but once it’s named, it’s named and the other is f-1. It doesn’t mean the functions f and f-1 are equal and interchangeable. It means that if they have/are inverses, there is a relationship. We’re investigating that relationship.

Applying Generalizations: Relationships between Functions and Inverse Relations (HANDOUT 4)

In this investigation, students apply their learning to the specific example they did previously (Handout 2). Before, they were reasoning through the problem, sense-making through trial and error and beginning to observe patterns based on initial knowledge. Now, they will apply the concepts they gained through generalization they generated in the last investigation (Handout 3), using composition to preview a new approach to verifying inverse functions, symbolically.

>>> Teacher note: Here, students are experiencing the Common Core rigor shift, through a combination of developing fluency with composing functions and applying conceptual understanding of composition to inverse functions, as well as making use of structure and reasoning to form and apply generalizations (SMP2 Reason abstractly and quantitatively; SMP7 Look for and make use of structure; SMP8 Look for and express regularity in repeated reasoning). This builds on the previous standard F-BF.4a and builds up to lesson 4. We’re working toward coming to the generalization, in symbolic notation, that f(f-1(x)) = x and f-1(f(x)) = x. Allow students to continue to discover and explore, by pushing their thinking and observations of structure and relationships

Conclude this investigation with a discussion about the observations and conclusions that students have gained thus far. Pull out important points from students’ comments that foreshadow the idea of composition and that they will learn a new method for verifying functions – symbolically. Discuss student ideas regarding the following topics (this is also a good opportunity for students to write a math journal reflection):

* The domain and range of functions and their inverses
* The tabular and graphical methods for verifying inverse functions
* Their own reflections on the difference between the way they solved the problem the first time (Handout 2) and the second time (Handout 4)

Why Verify?

Have a quick discussion to solidify the point about the need to verify inverse functions: (1) that the inverse relation is, indeed, the inverse and (2) that the inverse relation is, indeed, a function. We now have three methods – tabular, graphical, and symbolic via composition. Ask students to consider the questions below in a think-pair-share format, followed by a whole class discussion.

* What do we need to verify?
* Why do we need to verify?
* What are the strengths and limitations of each verification method? In other words, what information do we learn, and what information are we not able to gain, about the functions and their inverses through each verification method?

Composition and Composite Applications (HANDOUT 5)

*In previous courses, you have made new functions by combining 2 or more different functions, using mathematical operations such as addition, subtraction, multiplication, and division. We will now learn there is another way to build new functions from 2 or more different functions. This operation on functions called composition, which you can also call or think about as, “a function of a function.” It is an operation that combines functions. We will learn how to combine functions, so that we can use this method to continue to think about verifying inverse functions. As we explore, remember to pay special attention to domain and range.*

This introduction to composition is by reasoning and applying conceptual understandings gained by generalization (SMP2 Reason abstractly and quantitatively; SMP8 Look for and express regularity in repeated reasoning) in the previous investigations. Have students work individually first on problems 1-3, then share their ideas with a partner, and then continue to problem 4 by working together as a pair.

>>> Teacher Note: This investigation contrasts the traditional approach to teaching composition by blindly applying a procedure to compose one function with another. Students will learn to see functions as compositions of each other, beginning by decomposing one function into two separate functions. By flexibility with decomposing and composing, they will gain deeper understanding.

>>> Teacher Notes by question number:

Problem 2. Students should be aware that k is the square function and therefore (x – 3) must be positive, so x > 3.

Problem 3. Most students will say they used substitution, but have them be more specific to the operations they applied and the order in which they applied them. For example, “I would first compute (x-3) and then square root the result (2 steps).” Precision is important in communicating about and solidifying their understanding (MP6).

Problem 4. Sample solution: g(x) = x - 3 and f(x) = x 1/2

Formalize the process for students, allowing time to think and discuss. The following generalization should emerge:

*Since we know k(x) = (x-3)1/2 is it true that k(x) = [g(x)] 1/2? Why? Because g(x) = x - 3*

*Is it true to state k(x) = f(g(x))? Why? Because f(x) = x ½ so f(g(x)) = (x-3)1/2*

Students should be able to see, and discuss, that k has the same effect as g and f evaluated successively, beginning with g the ‘inner’ function g(x). Discuss the idea with students that we could say k is “composed” of two functions, f and g, where the output of the g function becomes the input for the f function. Ask students to express this in their own words, and to express what this means in terms of domain and range.

Making Sense of Composition (HANDOUT 6)

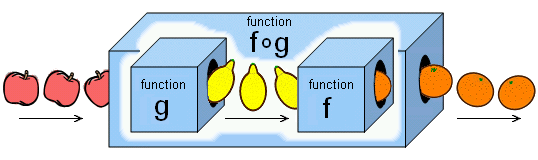
This exploration takes the concept and process of composition further, revisiting the strategy of decomposing and composing in order to make sense of the relationship and using an expanded pictorial model from the earlier activity, to develop the formal definition of composition. Start by taking a quick survey of the whole class on how many functions students see (1, 2, or 3). Tally the results, have students work in pairs on problems 1-3, and then re-do the vote to see if students have changed their ideas (before they move into problem 4).

>>> Teacher note: The key here is learning to see the composite function as 1 function by breaking it into 2 functions. Also, since the composition function is a new function, some students may say they see 3 functions. Both answers are correct; the differences lie in the explanation and reasoning. Students will see that each function h(x) and g and f are their own functions. Again, emphasize precision in language and notation as students communicate their observations and ideas; it will help them clarify their understanding (SMP2 Reason abstractly and quantitatively; SMP6 Attend to precision).

>>> Teacher Note: Domain f(g(x)) is set of all real numbers x such that f(g(x)) is in the domain of f, and output of f(g(x)) is number, actually in the domain. Domain f(g(x)) such that g(x) is in domain of f. This might feel strange at first, because students are putting the output of another function into the first function. They will have to consider the entire set of inputs and outputs (domain and range) as an entire set of values, which is an important distinction, very different from a single value x that falls within the set of values that define the domain or range.

Discussion/Summary Opportunities:

As you review students’ work as a class, ask details about the individual functions and the composite functions f(g(x)) and g(f(x)). Tell students: *We can see the composite functions as new functions, themselves. We can even name the new function itself, for example, k(x) = f(g(x)).*



>>> Teacher Note: The notation for a composite function uses a circle, as in the picture above. The Domain of k(x) is the set of x values for g that will allow g(x) to be in the domain of f. In the picture above, students see the input of g as apples and the output of g as lemons, which is also the input of f. The output of k(x) is the output of the composite function, f(g(x)).

Consumer Spending (HANDOUT 7)

This activity will help formalize the operation of composition. Now students will use a financial application of learning they gained by reasoning about composition. Ask students to answer only problems 1-3 on the handout. Students will create functions for each single discount (individually first), and then share their work with a partner.

Alternatively, students could begin this activity by first taking a position – ask students to first predict which option they believe is the better deal (one coupon or both coupons). Students could pair up with someone who made the same prediction, and work through problem 1 together, to confirm their prediction (or find out that their prediction was incorrect). Then, partners could break up and join a different partner, someone who initially made a different prediction than theirs, to share their work and either confirm or disprove their own and their partners’ reasoning. (SMP2 Reason abstractly and quantitatively; SMP3 Construct viable arguments and critique the reasoning of others)

Give students think time individually, and then have pairs discuss their reasoning. Share out different responses with the class. Look for student solutions that have taken into consideration the price before the discount (domain), as that may affect the “Best Deal.”

>>> Teacher note: Students might ask questions about price for items, but have them consider a variety of prices, because the goal is to reach some generalizations based on multiple examples (MP8). Answers will depend on their purchase. Students might also ask if they can get money back, if they used the $20 coupon for something that cost less than $20. Let the class discuss it, so that they can explore the math involved in the question. If students may also bring up the order of applying multiple discounts, ask them to consider and discuss: Is the order important? What order would they prefer? Why?

Discussion questions:

* If you could use only one discount, which would you use? Why?
* If you could use both discounts, in which order would you use them? Why?
* How did your ideas about the order in which you use the discounts change, as you progressed through the problem?

Summarize/conclude by having students consider what happens when we can use only one discount and when we can use both discounts in different orders, for example, problem 2 ($100 of clothes). Revisit problems 3-4 to gather students’ ideas about multiple examples and counter-examples, as well as their interpretation of the situation in context beyond just their calculations. Reflect on the problems as a class and formalize a generalization. Use any of discussion questions below, and others of your own.

>>> Teacher Note: Problem 4 is the formalization/generalization of composition function, which represents the combination of the two earlier functions. Students should recognize an important point about composition – composition is an operation, but it is not commutative. The order does matter!

*Annotated Answer Key for Consumer Spending Problem* (Handout 7)

|  |  |  |
| --- | --- | --- |
| Discounts | Functions | Composition |
| Example of buying $100 worth of clothes | For x being purchase price |  |
| c. Single discount 20% off  Ex. 100 -20 = **$80** | f(x) = x - .20x  ***f(x) = .8x*** | No composition needed. (f(x) can be shown to be the product of two functions .8 and x) |
| c. Single discount $20 off.  Ex. 100 – 20 = **$80** | ***g(x) = x – 20*** | No Composition needed. (g(x) can be shown to the difference of two functions x and 20) |
|  | | |
| d. Both discounts  20% off first then $20 off  Ex.   1. 100 -.20(100) = $80 2. $80 – $20 = **$60** | ***h (x) = .8x - 20*** | *20% 0ff first f(x) = .8x*  g(20% off) = g(f(x)) = g (.8x)  g(x) = x - 20  ***g(.8x) = .8x – 20 = g(f(x))*** |
| d. Both discounts  $20 off first then 20% off  Ex.   1. 100 – 20 = $80 2. 80-.20(80)= 80 – 16 = **$64** | ***k(x) = .8(x – 20)*** | *$20 off first g(x) = x – 20*  f($20 off)) = f (g(x)) = f(x-20)  f(x) = .8x  ***f(x - 20)= .8(x – 20) = f(g(x))*** |

Understanding Composition: Frayer Model (HANDOUT 8)

*Now we will literally create new functions using the “operation” of composition that we just explored.*

This activity will advance students’ understanding of the meaning of f(f -1 (x)) and f -1 (f(x)), in preparation for Lesson 4, where they will find the equation of an inverse function by solving for f-1 in the equation f(f -1 (x)) = x. Students have evaluated f(f -1 (x)) and f -1 (f(x)) for specific values of x; they will now formalizes their ability to evaluate values of functions for all x by writing a symbolic representation (equation) for f(f -1 (x)) and f -1 (f(x)).

Have students work on this model individually for the first section (“Numerically” section, see Handout 8), and then share their ideas in pairs (turn and talk). Then have them work together to share and listen to each other’s reasoning as they work on the rest of the sections in partners (MP2). Have them rotate through different partners for each subsequent section of the chart (total 4 partners).

>>> Teacher notes below, by section of the Frayer Model chart (Handout 8), include notes about possible solutions, as well as tips for discussion questions to guide students’ exploration and coming to deeper understandings together. Guiding questions embedded below could be posted on flipchart or projector or on a separate handout for reference while students work with different partners.

*Numerically*: Ask students to consider the functions f(x) = 2x and g(x) = x2. Ask what the numerical answer would be for f(g(x) if x=5? For x= -4? For X= √7? For X= a? Numerically they can first substitute 5 into g(x); g(5) = 25. Then they can evaluate f(g(5) = f(25) = 2(25) = 50. Similarly f(g(-4)) = f(16) = 32 and f(g(√7) = 14, f(g(a))= 2a2 .

*Algebraically*: Ask students to consider what f(x2) would look like. Have students explain why f(x2)= 2x2. Discuss the idea that they have created a new function that is **composed of** f(x) and g(x). We can call the new function h(x), for example; h(x) = f(g(x))= 2 x2.

x g(x) f(g(x))

**g**

**f**

*Inputs and Outputs*: Have students think about inputs and outputs of each separate function and of the new composed function. Have them communicate their ideas both in terms of input and output, as well as domain and range. They should see that f(g(x)) is a new function whose inputs, x, are inputs of the function g and whose outputs are the numbers f(g(x). (The Domain of the new function is the Domain of g, which will be discussed more fully later).

* Introduce a new way of communicating the meaning of what students are now seeing: *Functions can be “composed” when a portion of the range of the first function lies in the domain of the second.* Ask students to explain to each other, in their own words, what this statement means for f and g in this Frayer Model.
* Have students discuss the *order of a composition***:** We say f(g(x)) is the composite of g and f. It is made by composing g and f in the order g, then f.
* Ask students if f(g(x)) and g(f(x)) define the same function. Have them create an argument for their response and share with a partner, critique each other’s reasoning (MP3), then discuss with the whole class. Look for students who argue that finding a few values give different results; ask those students if they can prove it for all numbers. Students will see that the equation for g(f(x)) results in 4x2. This proves that g(f(x)) ≠ f(g(x)) and that the “operation” of composition is not commutative.

*Notation*: Make students aware that the conventional notation in textbooks for composition is a circle, ○. We read f ○ g as “f of g,” and the value of f ○ g for the value of x is noted as (f ○ g)(x) or f(g(x). (Similarly for g ○ f) Give them a few additional examples to try, to see if they can translate and state the meaning in words and notation. Especially for ELL students, be sure to have them write with notation and with words, and speak the notation and explanation. Re-emphasize to all students that composition is an operation in itself, and the parentheses or dots do not mean multiplication here.

>>> Teacher note: For a challenge question, foreshadowing the next lesson, ask students if domain and range are important when considering composition. Technically, it doesn’t matter when we’re simply exploring the mechanics of symbolic representations. However, this is an opportunity to build on that compelling reason for the need to verify inverse functions. Each verification method has limitations; this idea was introduced earlier. With composition (symbolic verification), simply finding the equations of the composite functions does not tell you whether the inverse function is, indeed, a function. Only through a consideration of domain and range, reinforced by the emphasis on the 1-to-1 correspondence, are we able to verify. These ideas lead into Lesson 4, in which students will learn how to make a function invertible even if it is not (i.e., restrict domains if we find, by exploration of domain and range after composition, that the inverse relations are not function).

Ask students questions about the domain and range of the composite functions g(f(x)) in the following example only.

Students should understand and be able to explain why the domain of g depends on the output of f.

**Single function f(x) Composite Function g (f(x))**

**Domain for f is the input x Domain for g is the Range of f or f(x)\***

**Range for f is the output f(x)**

A graphic showing a single function f(x).
The domain for f is the input x   
The range for f  is the output f(x)
a graphic showing Composite Function g (f(x)). THe
Domain for g is the Range of f or f(x)*


*Inference*: The last question encourages students to synthesize their thinking toward a generalization. By composing functions, the answer is, indeed, equal. This generalization is very significant, and will be used in the next lesson. Engage students in discussion about why, using multiple representations to justify their reasoning (SMP2 Reason abstractly and quantitatively; SMP8 Look for and express regularity in reasoning).

Technology: Investigating Composite Functions, Part 1 (HANDOUT 9)

Using a TI-84+ or comparable graphing tool that allows graphing composite functions, this investigation engages students in graphing a pair of functions f and g, as well as the composites g ○ f and f ○ g. They will make conjectures about the domain and range of the composite functions, and then test their conjectures by writing the equations for the composite functions.

For this investigation, have students work in pairs or groups of 3. To summarize the investigation as a class, discuss the key features of each of the original functions in comparison to the composite functions, especially with respect to domain and range. Invite students’ observations and conclusions.

>>> Teacher note: Problem 5 is an extension, simply uncovering the idea that composition does not only have to involve 2 functions.

Technology: Investigating Composition of Inverse Functions, Part 2 (HANDOUT 10)

Now students will investigate pairs of functions in the same manner as above, except that in Part 1, the composite functions were not inverses of the original functions. Here in Part 2, the composites *are* inverses. This exploration is designed to generate multiple examples to lead students toward strong generalizations about the composition of inverse functions (SMP7 Look for and make use of structure; SMP8 Look for and express regularity in reasoning) and now formally introduce a third method for verifying functions – symbolically, by composition. Students will, again, make predictions/conjectures about functions and their composites, including the domain and range, before actually finding the equation of the composite.

Problem 3 could be done as a group activity, by giving groups of 3 flipchart paper and asking students to “Write as many statements as you can about determining whether functions are inverses or not – include discussion of features including domain, range, graph itself, symbolic expression, representations, etc.).”

>>> Teacher notes by problem:

* In Problem 1, students have seen the natural log function in prior courses (Algebra 2 standard F-LE.4). It doesn’t matter how much they remember about solving and working with the natural log expression, because the goal is to reason, not to find the solution; students will use the calculator to aid their analysis (SMP2 Reason abstractly and quantitatively; SMP5 Use appropriate tools strategically).
* In Problem 2, the composite should be a straight line. Students should be recognizing, even predicting before completing all problems, the similarities in the graphs and equations of the composition of inverse functions (the composition generates the function f(x) = x and graphs as a straight line).
* In Problem 3, students cement their conclusions about the principle of inverse functions f(f-1(x)) = x and f-1(f(x)) = x. They should be interpreting the meaning of this significant generalization in terms of domain and range and the relationship between functions and their inverses.

**Lesson Closing**

Verifying Inverse Functions Symbolically(HANDOUT 11)

This exploration is an application of the learning students have gained on verification by composition. This could be conducted as a partner activity, as a rotation activity in which partners are assigned to 2 problems and a different set of partners verify each other’s their work, or it could be assigned as homework and reviewed the next day. It could also be used as a formative assessment to check for students’ understanding to this point. The reflections emerging from this practice work will be important to bring out as a class, regardless of the style of instruction you choose.

>>> Teacher notes, by problem:

* The goal is to have students practice the symbolic method of verification by composition (one of the standards being addressed in this unit F-BF.4.b), even though other methods could be used. The final question (problem 7) asks students to consider other methods, so that they can also reflect on the appropriateness of different methods for different situations.
* Problem 2 does not have an inverse.
* Problem 4 poses the potential misconception that has been surfacing throughout regarding notation. Make sure students understand by now that
* Problem 5 is a great opportunity to revisit the idea that when inverse functions are composed with each other, the result is always the same (x). Even though composition itself is not commutative, the composition of 2 inverse functions generates the same result (again, be precise, this does not mean that composition of inverse functions is commutative; it is still not commutative, just make sure students understand that the result is the same). Some students may realize that they can use the result to their advantage, since it does not matter which function we define as f vs. f-1. More specifically, students may be able to observe the structure of the equations of the functions (SMP7 Look for and make use of structure) and notice that it will be easier to square rather than to take the square root. Prompt students to discover this idea on their own by asking, “Which function do you want to define as f, and which as f-1? Why? And why does this matter, or does it matter with composition?”
* Problem 6 will require a logarithm to verify. This problem foreshadows Lesson 5.

Ticket to Leave

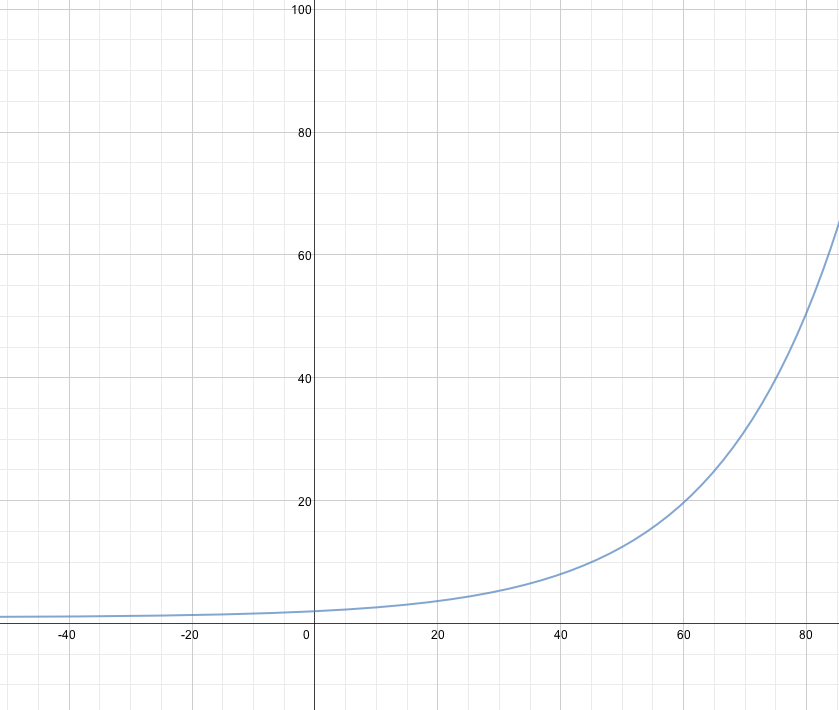
Use the following questions to conclude the investigation above (particularly problem 7), as a formative assessment:

* Which other methods might you have used? Which method is more appropriate for corresponding problems? Explain your reasoning.

**Lesson 3 Handout 1**

**Tabular/Numerical Investigation of Inverse Functions, Part 1**

The graph below shows the growth of an initial investment of $1 over time, where the range represents the value of the investment in dollars and the domain is the time in years.

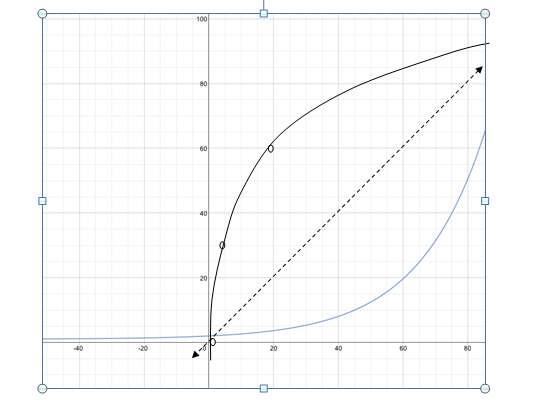


1. Using symmetry between a function and its inverse sketch a graph of the inverse.
2. Make 2 tables of values, one for the function and one for its inverse.
3. Shown on both of your graphs the point at which the initial investment doubled in value? Explain what that point means in the context of the problem.
4. What do you notice in the tables about the relationship between the function and its inverse?
5. Is the inverse a function? Why? Describe your reasoning.

**Lesson 3 Handout 1 ANSWER KEY**

**Tabular/Numerical Investigation of Inverse Functions, Part 1**

The graph below shows the growth of an initial investment of $1 over time, where the range represents the value of the investment in dollars and the domain is the time in years.



1. Using symmetry between a function and its inverse sketch a graph of the inverse.

*Answers should show symmetry in the line y = x*

1. Make 2 tables of values, one for the function and one for its inverse.

*Answers will be approximate based on graph. The two tables may contain identical values in reverse or may be random for both.*

1. Shown on both of your graphs the point at which the initial investment doubled in value? Explain what that point means in the context of the problem.

*Solution should be near the point (30, 5) for the original function which means it takes 30 years to go from an initial investment of $2.50 to a value of $5.00. The inverse graph point (5, 30) means $2.50 will double after 30 years of investing.*

1. What do you notice in the tables about the relationship between the function and its inverse?

*See answer 2 above.*

1. Is the inverse a function? Why? Describe your reasoning.

*Yes, it passes the horizontal line test or for each input there is exactly one output.*

**Lesson 3 Handout 2**

**Tabular/Numerical Investigation of Inverse Functions, Part 2**

Here are some of the conclusions we have been forming so far:

* Two graphs are inverses of each other if they are symmetrical about the line y = x.
* If a function is one-to-one, then the inverse is also a function.
* In an inverse relationship between two functions, the domain of the function becomes the range of the other and the range of the function becomes the domain of the other.

Now we will extend this to tables and determine how to verify that two functions are inverses using numerical/tabular representations.

1. The two tables represent ordered pairs defined by a function, f(x), and an inverse function, *f-1(x).* Complete the two tables below.

Table 1 Table 2

|  |  |
| --- | --- |
| *x* | *f-1(x)* |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
| -1 |  |
| -3 |  |
| -5 |  |
| -7 |  |
| -9 |  |

|  |  |
| --- | --- |
| *x* | *f(x)* |
| -2 |  |
| -1 |  |
| 0 |  |
| 1 |  |
| 2 |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

1. Use the tables above to answer the following questions.
   1. How can you determine? Which table did you use?
   2. How can you determine? Which table did you use?
   3. What does f-1 (x) mean?
   4. f(f-1(-7)) =
   5. f-1(f(2)) =

**Lesson 3 Handout 2 ANSWER KEY**

**Tabular/Numerical Investigation of Inverse Functions, Part 2**

Here are some of the conclusions we have been forming so far:

* Two graphs are inverses of each other if they are symmetrical about the line y = x.
* If a function is one-to-one, then the inverse is also a function.
* In an inverse relationship between two functions, the domain of the function becomes the range of the other and the range of the function becomes the domain of the other.

Now we will extend this to tables and determine how to verify that two functions are inverses using both numerical and tabular representations.

1. The two tables represent ordered pairs defined by a function, f(x), and an inverse function, *f-1(x).* Complete the two tables below.

Table 1 Table 2

|  |  |
| --- | --- |
| *x* | *f(x)* |
| -2 | ***-9*** |
| -1 | ***-7*** |
| 0 | ***-5*** |
| 1 | ***-3*** |
| 1.5 | ***-2*** |
| 2 | ***-1*** |
| ***-1*** | ***-3*** |
| ***-3*** | ***-11*** |
| ***-5*** | ***-15*** |
| ***-7*** | ***-19*** |

|  |  |
| --- | --- |
| *X* | *f-1(x)* |
| ***-2*** | ***1.5*** |
| ***-1*** | ***2*** |
| ***0*** | ***2.5*** |
| ***1*** | ***3*** |
| ***2*** | ***3.5*** |
| -1 | ***2*** |
| -3 | ***1*** |
| -5 | ***0*** |
| -7 | ***-1*** |
| -9 | ***-2*** |

1. Use the tables above to answer the following questions.
   1. How can you determine? Which table did you use?

You can find the x so that f(x) = -3 using table 1

* 1. How can you determine? Which table did you use?

You can find the value of f(2) from table 1

* 1. What does f-1 (x) mean?

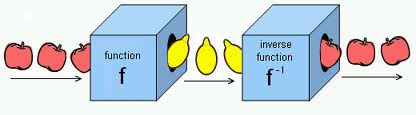
It means the value of the inverse function for input x

* 1. f(f-1(-7)) = f(-1) = -7
  2. f-1(f(2)) = f-1(-1) = 2

**Lesson 3 Handout 3**

**Generalization: Relationships between Functions and Inverse Relations**

Refer to the diagram.

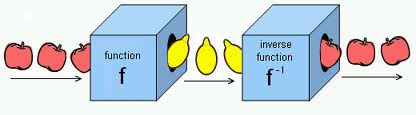


1. What do the lemons represent in this problem?
2. What do the apples represent in this problem? (two answers)
3. Complete the equations:
   1. .
4. Based on the diagram and the concept of inverse functions determine if the following equations are True or False:

**Lesson 3 Handout 3 ANSWER KEY**

**Generalization: Relationships between Functions and Inverse Relations**

Refer to the diagram.



1. What do the lemons represent in this problem?

*They are the ouput (range) of f. They are the input (domain) of f-1.*

1. What do the apples represent in this problem? (two answers)

*Apples represent the Domain of f and the range of f-1.*

1. Complete the equations:
   1. .
2. Based on the diagram and the concept of inverse functions determine if the following equations are True or False:
   1. *False. Because f and f-1 are inverses of each other and one to one functions the output of one function is the input of the other function and vice versa. So, f-1 (apples) = lemons and f (lemons) = apples also can define this inverse relationship.*
   2. False
   3. True
   4. True
   5. False
   6. False
   7. True
   8. True

**Lesson 3 Handout 4**

**Applying Generalizations: Relationships between Functions and Inverse Relations**

Refer to your learning from the Generalizations you made earlier (Handout 3).

1. Explain the statements: and
2. Let’s return to the linear function we started at the beginning and prove the finding from question 5 for

|  |  |
| --- | --- |
| *x* | *f(x)* |
| -2 | ***-9*** |
| -1 | ***-7*** |
| 0 | ***-5*** |
| 1 | ***-3*** |
| 1.5 | ***-2*** |
| 2 | ***-1*** |
| ***-1*** | ***-3*** |
| ***-3*** | ***-11*** |
| ***-5*** | ***-15*** |
| ***-7*** | ***-19*** |
| ***-9*** | ***-21*** |

|  |  |
| --- | --- |
| *x* | *f -1(x)* |
| ***-2*** | ***1.5*** |
| ***-1*** | ***2*** |
| ***0*** | ***2.5*** |
| ***1*** | ***3*** |
| ***2*** | ***3.5*** |
| -1 | ***2*** |
| -3 | ***1*** |
| -5 | ***0*** |
| -7 | ***-1*** |
| -9 | ***-2*** |

1. b.

c. d.

**Lesson 3 Handout 4 ANSWER KEY**

**Applying Generalizations: Relationships between Functions and Inverse Relations**

Refer to your learning from the Generalizations you made earlier (Handout 3).

1. Explain the statements: and

*The relationship of inverse functions is that they ‘undo’ each other and give back the original input of the function or the function inverse.*

1. Let’s return to the linear function we started at the beginning and prove the finding from question 5 for

|  |  |
| --- | --- |
| *x* | *f(x)* |
| -2 | ***-9*** |
| -1 | ***-7*** |
| 0 | ***-5*** |
| 1 | ***-3*** |
| 1.5 | ***-2*** |
| 2 | ***-1*** |
| ***-1*** | ***-3*** |
| ***-3*** | ***-11*** |
| ***-5*** | ***-15*** |
| ***-7*** | ***-19*** |
| ***-9*** | ***-21*** |

|  |  |
| --- | --- |
| *x* | *f -1(x)* |
| ***-2*** | ***1.5*** |
| ***-1*** | ***2*** |
| ***0*** | ***2.5*** |
| ***1*** | ***3*** |
| ***2*** | ***3.5*** |
| -1 | ***2*** |
| -3 | ***1*** |
| -5 | ***0*** |
| -7 | ***-1*** |
| -9 | ***-2*** |

1. -2 b. -2

c. 2 d. 2

**Lesson 3 Handout 5**

**Composition and Composite Applications**

1. Evaluate the function f(x) = 5x – 2x2 for the following values of x:
   1. f(3)
   2. f(3x)
   3. f(x-1)
   4. f(a)
2. Consider the function k(x) = (x-3)1/2 . What is the domain of the function k? Why?
3. Find the value k(7) and write down the process for finding k(x).   
   Discuss with a partner.
4. Create a rule (function) for each step stated above.   
   First work individually, then share with a partner.

**Lesson 3 Handout 5 ANSWER KEY**

**Composition and Composite Applications**

1. Evaluate the function f(x) = 5x – 2x2 for the following values of x:
   1. f(3) = 15-18=-3
   2. f(3x) = 15 x2 - 18 x3
   3. f(x-1)= 5x – 5 -2(x2 – 2x + 1) = -2 x2 + 9x - 7
   4. f(a) = 5a – 2a2
2. Consider the function k(x) = (x-3)1/2 . What is the domain of the function k? Why?

*X > 3 because k is a square root function*

1. Find the value k(7) and write down the process for finding k(x).   
   Discuss with a partner.

*K(7) = √4 = 2*

*First subtract 3 from 7. Second take the square root.*

1. Create a rule (function) for each step stated above.   
   First work individually, then share with a partner.

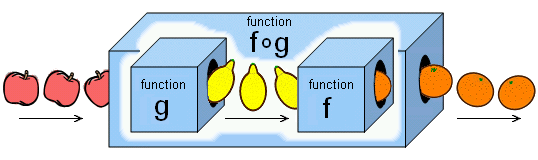
*First rule subtract 3 from a number f( x) = x – 3*

*Second rule take the square root of a number g(x) = √ x*

**Lesson 3 Handout 6**

**Making Sense of Composition**

This graphic visually represents an example of composition of functions.



1. How many functions do you see?
2. What is the Domain for each function?
3. What is the Range for each function?

**The Meaning of a Composite Function**

A composite function hcan be defined as the composite of the two functions fand gand denoted as h(x)=f(g(x))(read h of x *is equal to* f of g of x) or h(x)=(f\circ g)(x).

1. Given a function h(x) = ,
   1. Define two functions such that h(x) is the composition of f(x) and g(x). Write composition using function notation and justify your reasoning.
   2. What is the relationship between the two functions you defined and the original function h(x)?
   3. Predict, without calculating, the solution for g(f(x)). Explain your reasoning.

**Lesson 3 Handout 6 ANSWER KEY**

**Making Sense of Composition**

This graphic visually represents an example of composition of functions.

A graphic that visually represents an example of composition of functions.


1. How many functions do you see?

Three functions; f, g, and f ○ g

1. What is the Domain for each function?

The domain of f is lemons; the domain of g is apples; the domain of the composite f ○ g include the values from the domain of g and from the domain of f

1. What is the Range for each function?

**The Meaning of a Composite Function**

A composite function hcan be defined as the composite of the two functions fand gand denoted as h(x)=f(g(x))(read h of x *is equal to* f of g of x) or h(x)=(f\circ g)(x).

1. Given a function h(x) = ,
   1. Define two functions such that h(x) is the composition of f(x) and g(x). Write composition using function notation and justify your reasoning.

g(x) = x – 5, f(x) =

f(x) = and g(x) = x - 5 so f(g(x) = f (x-5) =

1. What is the relationship between the two functions you defined and the original function h(x)?

h(x)= f ○ g, h(x) is the composition of f with g

* 1. Predict, without calculating, the solution for g(f(x)). Explain your reasoning.

**Lesson 3 Handout 7**

**Consumer Spending**

Suppose you received two coupons for your favorite store: one that is a **20% discount**, and another one that is **$20 off**. You want to figure out which is the better deal if you are allowed only one coupon, or if the store allows you to use both coupons.

1. Explore the cost of buying a at least 3 different amounts of clothing (your choice), using a single coupon and using both coupons. What patterns are you noticing?
2. Calculate your cost of buying $100 worth of clothing…
3. Using a single coupon
4. Using both coupons
5. Define a function for each of the single coupons that will give you the discounted price for any purchase. Explain your reasoning.
6. Define a function for each way of using the multiple coupons that will give you the discounted price for any purchase. Explain your reasoning.

**Lesson 3 Handout 7 ANSWER KEY**

**Consumer Spending**

Suppose you received two coupons for your favorite store: one that is a **20% discount**, and another one that is **$20 off**. You want to figure out which is the better deal if you are allowed only one coupon, or if the store allows you to use both coupons.

1. Explore the cost of buying a at least 3 different amounts of clothing (your choice), using a single coupon and using both coupons. What patterns are you noticing?

|  |  |  |  |
| --- | --- | --- | --- |
|  | Cost with  20% off | Cost with $20 off | Cost with Both coupons  a) 20% off first b) $20 off first |
| Clothing $40 | .80(40)= $32 | $20 | 1. 32 – 20 = 12 2. 20 - .2(20) = 16 |
| Clothing $ 60 | .80(60)= $48 | $40 | 1. 48 – 20 = 28 2. 40 - .2(40) = 32 |
| Clothing $ 120 | .80(120)= $96 | $100 | 1. 96-20 = 76 2. 100- .2(100) = 80 |
| The more expensive items cost less if you use the 20% off coupon  Using the 20% off first then the $20.00 off is better than using the $20 off first. | | | |

1. Calculate your cost of buying $100 worth of clothing…
2. Using a single coupon

|  |  |  |  |
| --- | --- | --- | --- |
|  | 20% off | $20 off |  |
| Cost | .80 (100) = 80 | 100 – 80 = 80 |  |
|  |  |  |  |

1. Using both coupons

|  |  |  |  |
| --- | --- | --- | --- |
|  | 20% off then $20 off | $20 off then 20% off |  |
| Cost | .80 (100) = 80  80 – 20 = 60 | 100 – 80 = 80  .80(80) = 64 |  |
|  |  |  |  |

1. Define a function for each of the single coupons that will give you the discounted price for any purchase. Explain your reasoning.

20% off: f(x)= .80x $20 off : g(x) = x - 20

1. Define a function for each way of using the multiple coupons that will give you the discounted price for any purchase. Explain your reasoning.

f(g(x)) = f(x-20)= .8(x – 20) =.8x – 16

g(f(x)) = g(.80x) = .8x - 20

**Lesson 3 Handout 8**

**Understanding Composition: Frayer Model**

Use the following functions to complete the chart below:

**f(x) = 2x**

**g(x) = x2**

|  |  |
| --- | --- |
| **Numerically**  What is the meaning of the expression f(g(x))?  How would you find the value of f(g(x)) for x = 5? How about for x = - 4? x = √7? x = a ? | **Algebraically**  What is the meaning of f(x2)?  How could you write this expression in a different way? |
| **Notation and Vocabulary** | **Inputs and Outputs**  a graphic representation of the composition of two functions |
| **Inference**  Does f(g(x)) = g(f(x))? Why or why not? Explain your reasoning. | |

**Lesson 3 Handout 8 Understanding Composition: Frayer Model**

Use the following functions to complete the chart below:

**f(x) = 2x g(x) = x2**

|  |  |
| --- | --- |
| **Numerically**  What is the meaning of the expression f(g(x))?  It means the function f is evaluated using the value of g(x) as the input.  How would you find the value of f(g(x)) for x = 5?  *First find g (5) by squaring 5 then multiply the result by 2. (ans. 50)*  *How about for x = - 4? x = √7? x = a ?*  *Same procedure as for x = 5.*  *(Ans: 32, 14, 2a2 )* | ***Algebraically***  What is the meaning of f(x2)?  *Evaluate f using x2 as the input*  How could you write this expression in a different way?  *f(x2) = 2 x2* |
| **Notation and Vocabulary** | **Inputs and Outputs**  a graphic representation of the composition of two functions |
| **Inference**  Does f(g(x)) = g(f(x))? Why or why not? Explain your reasoning.  *No, f(g(x)) = 2x2 g(f(x)) = g(2x) = 4x2* | |

**Lesson 3 Handout 9**

**Technology: Investigating Composition of Functions, Part 1**

In this investigation, you will use a graphing calculator or online tool to analyze and make conjectures about a pair of functions and their composites.

**Let and g(x) = 1 – x**

1. Using your graphing calculator, enter the following functions:

Y2 = 1 – x Y3 = Y2(Y1(x)) Y4 = Y1(Y2(x))

1. Which of these functions correspond to f o g? Which corresponds to g o f?
2. Graph Y1, Y2, and Y3 (on your calculator) and sketch a rough picture of each graph below.
3. Make a conjecture about the domain and range of Y3.
4. Graph Y1, Y2, and Y4 (on your calculator) and sketch a rough picture of each graph below.

e. Make a conjecture about the domain and range of Y4.

2. Confirm your conjectures symbolically for parts 1c and 1e above, by writing equations for Y3 and Y4. Then find the domain and range of Y3 and Y4.

1. Y3
2. Y4

3. Composition is an operation that creates new functions from existing functions. Write an equation for the composition of functions Y1(Y2(Y3)), using the function Y3 that you determined above.

**Lesson 3 Handout 9 ANSWER KEY**

**Technology: Investigating Composition of Functions, Part 1**

In this investigation, you will use a graphing calculator or online tool to analyze and make conjectures about a pair of functions and their composites.

**Let and g(x) = 1 – x**

1. Using your graphing calculator, enter the following functions:

Y2 = 1 – x Y3 = Y2(Y1(x)) Y4 = Y1(Y2(x))

1. Which of these functions correspond to f o g? Which corresponds to g o f?

f o g =Y4  g o f = Y3

1. Graph Y1, Y2, and Y3 (on your calculator) and sketch a rough picture of each graph below.

*Do on Calculator*

1. Make a conjecture about the domain and range of Y3.

D: Various conjectures

R:

1. Graph Y1, Y2, and Y4 (on your calculator) and sketch a rough picture of each graph below.

e. Make a conjecture about the domain and range of Y4.

2. Confirm your conjectures symbolically for parts 1c and 1e above, by writing equations for Y3 and Y4. Then find the domain and range of Y3 and Y4.

1. Y3 = g( **) = 1 -**
2. Y4 = f(1 – x) =

3. Composition is an operation that creates new functions from existing functions. Write an equation for the composition of functions Y1(Y2(Y3)), using the function Y3 that you determined above.

1. Y1(Y2(**1 -**  )= y1 (1 - )**=**

**Lesson 3 Handout 10**

**Technology: Investigating Composition of Inverse Functions, Part 2**

In Part 1 of this investigation (Handout 9), f and g were not inverse functions. In Part 2, you will investigate pairs of inverse functions and their composites.

1. The pairs of functions below (P, Q, and R) are inverse functions.  
   For each pair, use your graphing calculator to complete and reason about the questions below.   
   **Be sure to sketch your graphs below** (in addition to finding them on the calculator).
2. Make a prediction about the features of the graph of the pair of functions. Explain your reasoning.
3. Graph each pair of functions on the same axes, and sketch the graph below. Were your predictions correct? Why or why not?
4. Make a conjecture about the graph of **f ○ f-1**, including what you expect for domain and range
5. Graph **f ○ f-1** to see your results. Were your predictions correct? Why or why not?
6. Make a conjecture about the graph of **f-1 ○ f**, including what you expect for domain and range
7. Graph **f-1 ○ f** to see your results. Were your predictions correct? Why or why not?

**Pair P**

f(x) = x3 and f-1 (x) = x 1/3

**Pair Q**

f(x) = 5 - 4x and f-1 (x) = (x-5)/4

**Pair R**

f (x) = ex and f-1 (x) = ln (x)

1. Verify your results (from above) symbolically, by finding the composition of f ○ f-1 and f-1 ○ f (for each pair of functions P, Q, and R). How do the composite functions confirm your findings, or not? Explain your reasoning.
2. Make a generalization about the difference between the composites of a pair of inverse functions and the composites of any pair of functions. Explain your reasoning.

**Lesson 3 Handout 10 ANSWER KEY**

**Technology: Investigating Composition of Inverse Functions, Part 2**

In Part 1 of this investigation (Handout 9), f and g were not inverse functions. In Part 2, you will investigate pairs of inverse functions and their composites.

1. The pairs of functions below (P, Q, and R) are inverse functions.  
   For each pair, use your graphing calculator to complete and reason about the questions below.   
   **Be sure to sketch your graphs below** (in addition to finding them on the calculator).
2. Make a prediction about the features of the graph of the pair of functions. Explain your reasoning.
3. Graph each pair of functions on the same axes, and sketch the graph below. Were your predictions correct? Why or why not?
4. Make a conjecture about the graph of **f ○ f-1**, including what you expect for domain and range
5. Graph **f ○ f-1** to see your results. Were your predictions correct? Why or why not?
6. Make a conjecture about the graph of **f-1 ○ f**, including what you expect for domain and range
7. Graph **f-1 ○ f** to see your results. Were your predictions correct? Why or why not?

**Pair P**

f(x) = x3 and f-1 (x) = x 1/3

**Pair Q**

f(x) = 5 - 4x and f-1 (x) = 

**Pair R**

f (x) = ex and f-1 (x) = ln (x)

1. Verify your results (from above) symbolically, by finding the composition of f ○ f-1 and f-1 ○ f (for each pair of functions P, Q, and R). How do the composite functions confirm your findings, or not? Explain your reasoning.

*P;Q;R: f ○ f-1 = x and f-1 ○ f = x*

*Predictions:*

*a)graphs of Functions are symmetric about the line y = x for all pairs*

*c) and e) Conjectures: the graph of f ○ f-1 and f-1 ○ f are the line y = x for all pairs*

1. Make a generalization about the difference between the composites of a pair of inverse functions and the composites of any pair of functions. Explain your reasoning.

*Composites of inverse functions are always the function f(x) = x because each “undoes” the operations of the other.*

**Lesson 3 Handout 11**

**Verifying Inverse Functions Symbolically**

Determine, using composition, whether the following pairs of functions are inverses of each other.

1. and
2. and
3. and
4. and
5. and
6. and
7. We investigated three methods for verifying whether a relation is the inverse of a given function – symbolic, graphical, and tabular. In the problems above, you used the symbolic method by composition. Choose any pair of functions (from problems 1-6) for which you would have preferred to use a different method to determine whether the pairs of functions are inverses of each other. Explain your choice and how a different method might be appropriate.

**Lesson 3 Handout 11 ANSWER KEY**

**Verifying Inverse Functions Symbolically**

Determine, using composition, whether the following pairs of functions are inverses of each other.

1. and

g(h(x)=2/(2/x +1 -1) = 2/(2/x) = x  *YES*

1. and

J(k(x)) = ( - 2)2 + 4 ≠ x if x = 5 for example *NO*

1. and

n( = ≠ x if x = 4 for example *NO*

1. and

f(3/x – 2) =( 2 + )/ 3 = 1/x  *NO*

1. and

*YES*

1. and

*NO*

1. We investigated three methods for verifying whether a relation is the inverse of a given function – symbolic, graphical, and tabular. In the problems above, you used the symbolic method by composition. Choose any pair of functions (from problems 1-6) for which you would have preferred to use a different method to determine whether the pairs of functions are inverses of each other. Explain your choice and how a different method might be appropriate.

# Lesson 4 – Restricting Domains of Non-Invertible Functions

**Time (minutes):** two 60-minute periods

**Overview of the Lesson**

In prior lessons, students have analyzed and verified whether graphical, tabular, and symbolic representations of inverse relations were, indeed, functions and their inverses. In this lesson, students learn to actually *find* the symbolic representation (equation) of an inverse function (given a function that has an inverse, i.e., an invertible function). They discover this by applying the principle of inverse functions discovered in Lesson 3, to solve for an inverse function. This approach develops deep meaning of the concept of inverse, providing a departure from the traditional approach of switching x’s and y’s. The strategy of creating an invertible function from a non-invertible function is also introduced (by restricting the domain of the function), with the goal of solidifying students’ understanding of identifying a unique inverse function.

As you plan, consider the variability of learners in your class and make adaptations as necessary.

**Standard(s)/Unit Goal(s) to be addressed in this lesson:**

**F-BF.B.4.d** (+) Produce an invertible function from a non-invertible function by restricting the domain.

**SI.3** Evaluate investment alternatives.

**RCA-ST.CS.4** Determine the meaning of general academic vocabulary as well as symbols, key terms, and other domain-specific words and phrases as they are used in a specific or technical context relevant to grades 11-12 texts and topics.

**SMP2** Reason abstractly and quantitatively

**SMP3**  Construct viable arguments and critique the reasoning of others.

**Essential Question(s) addressed in this lesson:**

Do all functions have inverses?

**Objectives**

* Find the symbolic representation (equation) of an inverse function, using the principle of inverse functions f(f-1(x) = x and   
  f-1(f(x)) = x
* Restrict domains to produce invertible functions, and analyze their graphs for end behavior
* Recognize the uniqueness of an inverse function

**Language Objectives**

* Connect root words and prefixes to meanings of words, specifically, “invertible” to “inverse” and non-invertible to “no inverse”

**Targeted Academic Language**

Invertible, non-invertible

**What students should know and be able to do before starting this lesson**

* Verify symbolically, by composition, whether a function is the inverse of an original function (F-BF.4.b)
* The principle of inverse functions, specifically, that f(f-1(x)) = x and f-1(f(x)) = x

**Anticipated Student Pre-conceptions/Misconceptions**

Students have been accustomed to solving equations for many years; however, their prior work has centered on finding a single solution, either represented by a set of numerical values or by variables. In this lesson, students will apply the same solution methods, using properties of equality, but their solution will actually be a function. Some students may be comfortable performing the symbolic manipulations required to get there, but they may have difficulty visualizing and conceptualizing the solution as an actual relationship (a function) rather than a set of specific values.

**Instructional Materials/Resources/Tools**

Graphing calculator or online graphing tool

**Instructional Tips/Strategies/Suggestions for Teacher**

Instructional notes are embedded in the Lesson description below, indicated by the following symbol: >>>

**Assessment**

* Finding the Inverse Symbolically (Handout 1) (“Synthesis” reflection, see Lesson 4 Lesson Details)
* Exit Ticket (formative assessment)

\* Items marked with asterisks (\*) above may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.

**Lesson Details (including but not limited to:)**

**Lesson Opening**

From Verifying to Finding

Engage students in an opening activity to activate prior knowledge and pose a new question about transferring that knowledge. Pose the following question: ***Can we find the inverse function of f(x) if we know the inverse exists?***

**Then ask: *If we have a non-invertible function f(x), might there be any way to find the unique input for an output?*** Gather a quick count of the number of students who respond yes and the number who respond no. Then have students pair up and share their reasoning with each other. Ask students, as a whole class, if anyone wants to change their vote after listening to their partner’s reasoning. Tell students, *There is a way, and we will focus on that today, but first, we need to be able to find the equation of the inverse function symbolically.*

Briefly discuss students’ recollection of important definitions/generalizations. Have students work in groups of 3, with each member diving into 1 of the statements below. Each member writes as many relevant and true pieces of information they can make about their statement (include graphical, tabular, and symbolic representations, as well as examples) on a separate piece of paper. Then they rotate their papers to the next person once, and then a third time. Each time they rotate, students add their own ideas about the statement. The fourth time, they have a group discussion about their conclusions for each statement.

* A function is a rule that assigns 1 output for each input. A function that assigns 1 input for each output is also called 1-to-1. What does that say about domain and range?
* A function that has an inverse function is called an “invertible” function. Invertible functions are one-to-one. What does that mean?
* What principle of inverse functions did we learn from composition? What does it mean?

*In lesson 2, we saw that the inverse relation of a function may or may not be a function. We found inverses graphically, observed the behavior of the inverse graphically, and determined if the inverse was a function. We also investigated what kinds of functions have inverses that are functions (quadratics did not). In this lesson we, will restrict the domains of functions that are not one-to-one to make them invertible and we will find the rule (equation) of their inverse functions.*

*In lesson 3 we saw that the composition of inverse functions led to the result of x: f-1(f(x)) = x = f(f-1(x)). What does this mean? In this lesson we will use that principle of inverse functions to find the rule (equation) of the inverse of one-to-one functions.*

***This lesson investigates the following questions: Can I find the inverse function of f(x) knowing the inverse exists? If I have a non-invertible function f(x), can I ever find the unique input for an output? (If so, then it is invertible).***

**During the Lesson**

Finding the Inverse of a Function: Three Methods

Lead students in an open lecture, in which you summarize important principles and ideas, and ask them questions along the way. Teacher notes are embedded with student instruction, below.

***Method 1: Backward Reasoning***

*One way to find the inverse of a function is to reverse the operations on the value of x that it specifies.*

>>> Teacher note: This may be familiar to students who found inverses of simple functions in Algebra 2. It is based on the principle of undoing or reversing the operations in the reverse order. They have also done similar reasoning in prior lessons in this unit. (MP2) This method is also called “back mapping,” which students may have learned in middle school with a flowchart for solving equations.

*Example: 2x + 1*

*Operations specified for any input value x:*

*1) Multiply x by 2*

*2) Add 1*

*So, starting with an input of x = 3, first multiply by 2, which gives us 6, then add 1, which gives an output of 7.*

*Finding the inverse expression means starting with 7 (the output) because we want to end up with 3, and reversing the operations.*

*1) Subtract 1*

*2) divide by 2*

*We will get the output 3.*

*So we can define the function f(x) = 2x + 1 and define the inverse function f-1(x) = (x - 1)/2.*

Discussion prompts for Method 1:

* What is the function f(x) and how would we define the inverse function f-1(x) in this example?
* How would you express this backward reasoning method, for this example, using the language of domain and range?
* Can we confirm f(x) and f-1(x) are inverse functions? How?

*For the third question, one way is to verify f(a) = b and f-1(b) = a.*

*Try using a particular value for a:*

*f(5) = 2(5) + 1 = 11 f-1(11) = (11 – 1)/2 = 5*

*But we don’t know that this is true for all values of x. In Lesson 3 we saw that the composition of a function and its inverse resulted in x. What does that mean? If we substitute any value x into f(x) (or another way to say this is if we evaluate f(x) for any value of x), and then take the result and substitute it into f-1(x), we will always get x back.*

Ask students to reflect on that statement:

*Is this always true? How can we be confident? Is it true for the composition of any two functions? (no) What did we learn in the prior lesson?*

Ask students to verify their example inverse functions via composition.

Example: f-1(f(x)) = f-1(2x+ 1) = [(2x + 1) – 1]/2 = (2x)/2 = x

*If two functions are inverses of each other this principle works both ways.*

Now have students verify that f(f-1(x)) = x.

Next, have students investigate expressions such as 3x3 – 1 and 3(x – 1)3 and describe the operations performed on x. Have them create a rule (expression) that will ‘undo’ the operations of the original expression.

>>> Teacher note: Encourage students to use a visual and/or pictorial representation to represent this backward reasoning approach, this time not for the purpose of aiding their reasoning, because this method should be familiar. Instead, having them think through a different lens is for the purpose of building fluency in thinking flexibly forwards and backwards, and also to drive the point home about the composition of inverse functions producing the x value backwards and forwards.

***Method 2: Interchanging x and y***

*This approach may also be familiar from prior courses, which most likely would have been introduced as a procedural method – simply flip the x and y and solve for y. Does this sound familiar? However, now that we have been exploring inverse functions and meaning of inverse in more depth in this unit, what do we now know about why this approach works? What is the conceptual understanding underneath?*

Example: Suppose f(x) = -3x + ½, find f-1(x).

Rewrite the equations using y = f(x): y = -3x + ½.

Then interchange the variables x = -3y + ½

Then solve for y:

X – ½ = -3y

(x – ½)/-3 = y

Then replace y with f-1(x)

(x – ½)/-3 = f-1(x)

>>> Teacher note: This approach may be familiar to students who found inverses of simple functions in Algebra 2 by solving an equation by flipping the x and y in an equation, and solving for y. This approach is less intuitive and more procedural. However, the reasoning and conceptual understanding behind the method is now accessible to students. Engage them in a discussion that draws on their prior learning in this unit. Points that should emerge include the symmetry of the graphs of inverse functions and their reflection about the line y = x, the pictorial and symbolic representations of functions and their inverses showing the output of the original function becoming the input of the inverse function, the nature of the domain and range in a 1-to-1 function and its inverse works both ways (domain becomes range and range becomes domain for every value 1-to-1), and students may even draw on their recent work in Lesson 3 on composition of inverse functions generating the value of x (only with inverse functions).

***Method 3: Solving the Composition Principle***

*Now we will apply our understanding of composition to actually find the inverse function. This is a symbolic approach using the composition principle of inverse functions. What was that principle and how did we find it?*

>>> Teacher note: This method introduces a new approach, based on the prior lessons in this unit. Now students will pull together their conjectures and generalizations from this unit, and their work in unfolding the meaning of inverse and verifying inverse functions through composition. They will apply their learning to this third method, a more conceptual method, to find the inverse symbolically through reasoning about composition and the meaning of inverse functions, rather than blindly applying procedures.

Ask students to recall and consider the principle of inverse functionsf(f-1(x)) = x or f-1(f(x)) = x, and also f(g(x)) = x. Have them “turn and talk” to a partner and discuss the following questions:

* Knowing this principle, how might you solve for g(x)? How might you solve for f-1(x) in the first equation? Or f(x) in the second equation?
* What would the solution to these equations mean?
* How is a solution to these equations different from the solution(s) of equations they have solved in prior courses?
* Try an example to help you consider these questions. Suppose f(x) = x3 – 5, find f-1(x) using the property f(f-1(x)) = x.

Generate discussion among groups, and then share out with the whole class, about the meaning of a solution to these equations.

*Possible solution for example* f(x) = x3 – 5:

f(f-1(x)) =( f-1(x)) 3 – 5. So we need to solve the equation (f-1(x)) 3 – 5 = x for f-1(x).

(f-1(x)) 3 – 5 = x

(f-1(x)) 3 = x – 5

f-1(x) = or

Application/Challenge Question: Ask students to justify this solution using Method 1, the backward reasoning method.

>>> Teacher note: Some students may have difficulty with the abstractness of this notion, whereas they previously solved equations for a numerical value or a variable x or a variable expression, they are now solving for a function. Symbolically, they are applying all of the same procedures, drawing on the properties of equality to isolate the item for which they are solving, and using inverse operations to find the solution. Except now, the solution is not a value, it is a function. A function has infinite values. This idea is deep. There is no need to solidify it just yet; this is just initial discussion. Next, students will investigate further with more examples.

Finding the Inverse Symbolically (HANDOUT 1)

Have students work in pairs on this exploration. Assign half of the teams in the class problems 1 and 2, and half of the teams problems 3 and 4. Once they are complete, have teams switch and verify each others’ work (teams that did problems 1 and 2 review another team that did problems 1 and 2; teams that did problems 3 and 4 review another team that did 3 and 4), and then have students return to their original teams to review the critique. (SMP2 Reason abstractly and quantitatively; SMP3 Construct viable arguments and critique the reasoning of others)

Have teams share out their conclusions (a representative from each group could come up to the board and post their work, or post on docu-camera or projector). In particular, when students report out, ask them to share not only their solutions, but also:

* What generalizations did you bring to their work that informed your thinking?
* How did the work of the team that reviewed your team’s work either confirm or make you think differently about your team’s original reasoning?

>>> Teacher note: For a function that is one-to-one, ask students how they found the inverse of a function from a graph or a table in previous lessons (Lessons 1 and 2)? Students may say it is easy to find its inverse from a table or a graph by interchanging the x and y-values. Next ask, “Can we find the inverse function rule if we have the symbolic form of the function?” Listen to students’ responses and then continue with advising them that there are several approaches to take. The goal is to get students thinking; you will formalize the next steps in the process.

>>> Teacher note: Question 4 is not invertible and can be used to segue into the next part of the lesson.

Students may continue to complete the second two problems that they did not do yet with their team, and conduct another round of review and critique, or they may complete remaining problems homework, to be revisited for further discussion on the next day.

Synthesis Reflection:

Synthesize this activity by having teams share their results and respond to the following question on an index card, which you can collect as a formative assessment and check for students’ understanding, as well as preview the next investigation:

* Summarize the three approaches for finding inverse functions.
* Provide 1 example for the composition method.
* Problem 5: Which of the functions were invertible? Explain your reasoning.

Invertible or Non-Invertible (HANDOUT 2)

Students learned in Lesson 3 that not all inverses of functions are, themselves, functions. In Handout 1, we used the term “non-invertible” function. Ask students to describe the reason for the name. Have students, especially ELL, make the connection between the root words “inverse” and “invert.”

Students should individually complete Handout 2 on deciding which functions are invertible or non-invertible. Then they will compare their responses with a partner and come to consensus. In order to reach consensus, they should each express their own reasoning, listen to each other’s reasoning, re-state each other’s reasoning, and then express whether they agree or disagree and why. They can pursue further discussion as needed. (MP3)

>>> Teacher note: Partners should discuss why each of the functions (considered one at a time) are non-invertible. Students should be thinking about in function A, for example, that for each of the output values (range), there are two input values (domain), with the exception of (0,0). Similarly in F and G, but in function H, students should see more than two outputs for some inputs.

*Solution:*

Students should recognize that the functions that are non-invertible are those that are not one-to-one (A, F, G, H).

>>> Teacher note: From prior lessons, students should know they can always find the inverse of a one-to-one function without restricting the domain of the function because a one-to-one function adds the requirement that each output (range) is linked to just one input (domain). For example, for a function that contains the points (1, 4), (2, 4), and (3, 4), the three points would not be points on the graph of a one-to-one function because 4 links to different numbers in the domain.

>>> Teacher note: Students should be able to identify a one-to-one function from its graph by using the Horizontal Line Test to mentally scan the graph with a horizontal line; if the line intersects the graph in more than one place, it is not the graph of a one-to-one function. If the function is not one-to-one, then its inverse will not be unique. Ask students to discuss what it means for the inverse function to be “unique.”

*We have discussed invertible and non-invertible functions, and we have begun to consider what it means for a function (or an inverse function) to be unique. Remember how we have emphasized the consideration of domain and range in our ongoing exploration of inverse functions. If an inverse of a function is not unique, there is a way to make it unique. In order to make the inverse of a function unique, the domain of the original function must be restricted. In this part of the lesson, we will investigate what it means for an inverse function to be unique and what it means to restrict a domain.*

Ask students, “Since being able to find input from output is important, is there a way to find an inverse function for *some* values of a non-invertible function?”

In pairs, ask students to consider which input values (values of the domain) in function A (revisit Handout 2), if eliminated from the domain, would result in a one–to-one function. In p, for example, they could limit the domain to x **>** 0 or instead x **<** 0 (Similarly for F, G, and H). Tell students it is possible to find the unique inverse for just that part of the graph.

Restricting Domains, Part 1 (HANDOUT 3)

In pairs, have students use their graphing calculators to investigate function *P* (Handout 3), to give them an opportunity to apply their learning to discover what it means to restrict domains in order to make non-invertible functions invertible. Because Part 2 (Handout 4) of this investigation is similar, you may wish to use Part 1 as a whole class to guide students through the process.

Students should recognize that this is the graph of a parabola. Model the process for making non-invertible functions invertible:

*Since a parabola is "U" shaped, it does not pass the horizontal line test and is not one-to-one. By graphical inspection, this function is not invertible.*

*By looking at the graph, we know that if we divide the x-axis into two parts at the point x = 3 and draw the dotted line x = 3, the left side of the graph would be one-to-one, and the right side of the graph would also be one-to-one, if each side were a separate function. The part of the x-axis to the right of x = 3 is the set of all real numbers in the interval [3,∞), and the part of the x-axis to the left of 3 is the set of all real numbers in the interval (-∞, 3].*

Have students follow along with the investigation, and show the graphing calculator (or online graphing tool) functions to make the function invertible (steps a-e in problem 1).

>>> Teacher note: Students should be able to graph the function via symmetry in the line y = x, by finding points on the graph and reversing the x and y coordinates, by using Drawing function on a calculator. They can then find the inverse function symbolically by using the composition principle for inverse functions to solve for f-1(x), by reasoning backwards, or by interchanging x and y and solving for by composition. Obviously, the composition method is recommended. Encourage students to explain their reasoning throughout the process (MP2) and associate their steps with the meaning of what they are actually finding.

Be sure to have students verify their results. See what ideas they have about doing this, first. Then guide students in a discussion of the process:

*Let’s check our results by choosing a value in the domain and evaluate the function f(x). Then substitute that result into f-1(x) (in other words, evaluate f-1(x) for the result you just obtained). What do you notice?* (They should get back the number they chose from the domain).

Restricting Domains, Part 2 (HANDOUT 4)

This investigation is similar, with a different function. Get students started on this problem by asking them to find the vertex (3, -2). They should only be interested in the x-value of the point, x = 3. Have them work in pairs on the similar investigation.

*Solution for finding the inverse symbolically:*

*f(x) = (x-3)2 -2, so on the interval* [3,∞), *f(x) is one-to-one and has an inverse.*

f(f-1(x)) = x

(f-1(x) – 3)2 –2 = x

(f-1(x) – 3) = ± (x + 2)1/2

so f-1(x) = 3 ± (x + 2)1/2

>>> Teacher note: The inverse is not unique because of the square root. There are two inverses, but only one will become the inverse of f-1(x) = 3 + (x + 2)1/2 or f-1(x) = 3 - (x + 2)1/2. It depends on what students choose as their restricted domain.

* If students chose the restricted domain to be [3, ∞), the inverse is f-1(x) = 3 + (x + 2)1/2 because the range of the inverse is equal to the restricted domain of the original function.
* If students chose the restricted domain to be (-∞, 3], the inverse is f-1(x) = 3 - (x + 2)1/2 because the range of the inverse is equal to the restricted domain of the original function.

**Lesson Closing**

Exit Ticket

Find the inverse of the functions below. Restrict the domains as needed. Show your process, the graph of the inverse function, and explain your reasoning.

1. f(x) =
2. f(x) = 3x2 – 4

**Lesson 4 Handout 1**

**Finding the Inverse of a Function Symbolically**

In Lesson 2 you used your graphing calculator to observe the graphs of the inverses of the functions below. Now, for the same functions (below), complete the following:

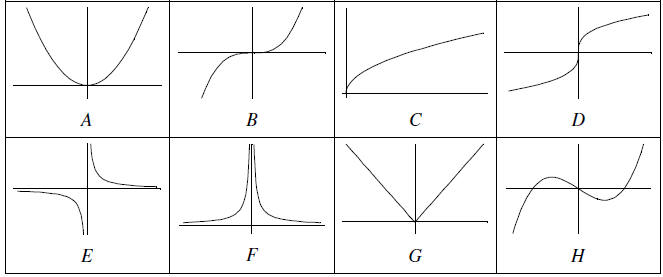
1. Write an equation for the inverse using the composition method, using the equation f(f-1(x)) = x
2. Write an equation for the inverse using any other method of your choice. Explain why you chose that method.
3. State the Domain and Range of the inverse.
4. Determine whether the Inverse is a function.

5. Which of the functions above were invertible? Explain your reasoning.

**Lesson 4 Handout 2**

**Invertible or Non-Invertible**

1. Determine which of the following functions are invertible and which are non-invertible. Justify your conclusions with supporting reasoning and evidence.



**Lesson 4 Handout 3**

**Restricting Domains, Part 1**

1. The function *P* below has its domain on the interval (∞, ∞).

a. Use your graphing calculator to find the inverse of the function.

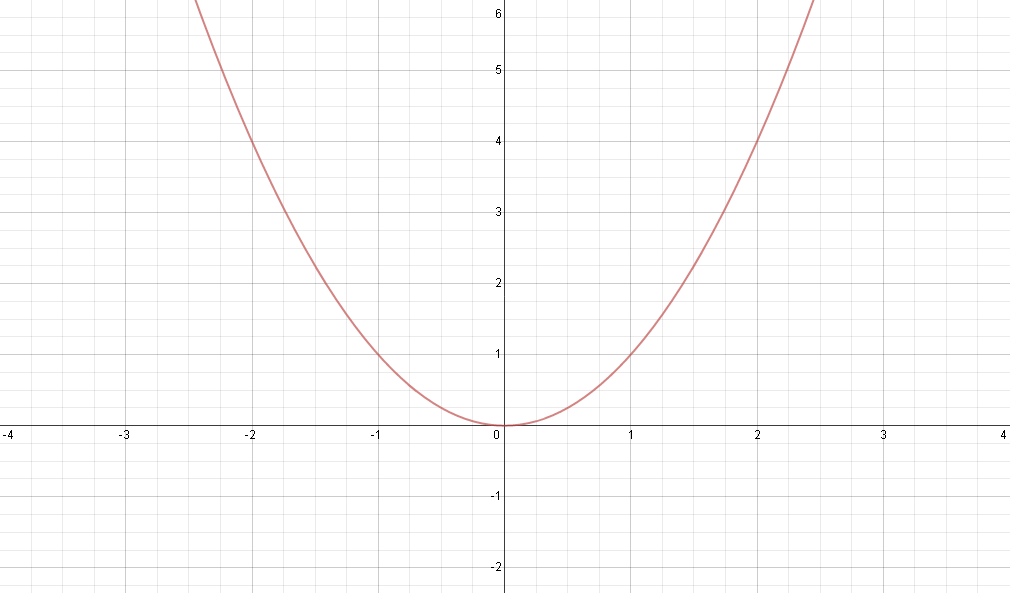
b. Find the symbolic representation (equation) for the inverse function using the composition method.

c. Verify the inverse function symbolically, using composition.

d. Is the inverse function unique? Make a unique inverse function by restricting the domain.

e. Verify that the function you created is now a unique inverse function.

*P*



**Lesson 4 Handout 4**

**Restricting Domains, Part 2**

1. The function *Q* below has its domain on the interval (∞, ∞).

a. Use your graphing calculator to find the inverse of the function.

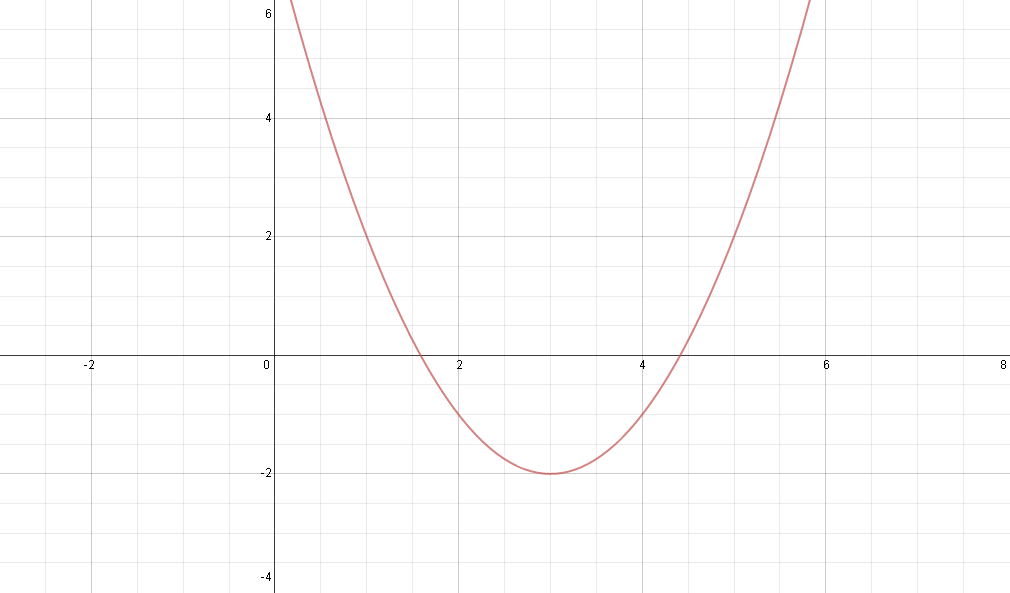
# b. Find the symbolic representation (equation) for the inverse function using the composition method.

c. Verify the inverse function symbolically, using composition.

d. Is the inverse function unique? Make a unique inverse function by restricting the domain.

e. Verify that the function you created is now a unique inverse function.

*Q*



# Lesson 5 – Applications of Logarithmic Functions

**Time (minutes):** two 60-minute periods

**Overview of the Lesson**

This lesson builds upon students’ prior knowledge of logarithms in Algebra 2 standard F.LE.4 (i.e., as expressions for the solutions to exponential equations/models). Now, students will formalize the meaning of a logarithmic *function* as the inverse of an exponential *function*. Using the principle of inverse functions, f(f-1(x)) = x and f-1(f(x)) = x, students will find the inverses of both exponential functions and logarithmic functions. They will revisit financial applications and apply their learning to solving investment problems symbolically, especially for *time (t)*, whereas previously they were limited to solving with graphical or tabular representations.

As you plan, consider the variability of learners in your class and make adaptations as necessary.

**Standard(s)/Unit Goal(s) to be addressed in this lesson:**

**F-BF.B.5** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

**FRDM.4** Make financial decisions by systematically considering alternatives and consequences

**SI.3** Evaluate investment alternatives.

**RCA-ST.CS.4** Determine the meaning of general academic vocabulary as well as symbols, key terms, and other domain-specific words and phrases as they are used in a specific or technical context relevant to grades 11-12 texts and topics.

**SMP2** Reason abstractly and quantitatively

**SMP4** Model with mathematics

**Essential Question(s) addressed in this lesson:**

How can functions and their inverses serve as tools to analyze financial situations?

How does the concept of inverse take on new meaning in the context of exponential and logarithmic functions?

**Objectives**

* Apply the principle of composition of inverse functions to exponential and logarithmic functions
* Recognize the inverse relationship between exponential and logarithmic functions, and define a logarithmic function in terms of an inverse function (of an exponential function)
* Interpret and model the inverse relationship between exponential and logarithmic functions in the context of real-world financial problems

**Language Objectives**

* Interpret the context of a real-world financial situation to determine a mathematical model, specifically, which part of the problem indicates the inverse (logarithmic) function

**Targeted Academic Language**

Tier 3 terms: Exponential function, logarithmic function (prerequisites: natural logarithm, common logarithm)

Tier 2 terms: “if and only if”

**What students should know and be able to do before starting this lesson**

* F-LE.4 For exponential models, express as a logarithm the solution to abct = d; evaluation the logarithm; for bases 2, 10, or e.
* The principle of inverse functions, f(f-1(x)) = x and f-1(f(x)) = x
* Familiarity with exponential functions from prior courses

**Anticipated Student Pre-conceptions/Misconceptions**

* Although exponential functions should be familiar, students’ prior exposure to logarithms was through the perspective of symbolic manipulation of logarithmic expressions and via the definition of a logarithm as the solution to the equation abct = d. Two ideas in this lesson will be new, and potentially difficult to grasp, at first: (1) seeing a logarithm as a function and (2) seeing the logarithmic function as the inverse of an exponential function.
* Although it should be familiar from prior courses, students often get confused with the notation for logarithmic functions.

**Instructional Materials/Resources/Tools**

* Graphing calculator or online graphing tool
* “Saving for Retirement” Video from PBS Learning Media http://mass.pbslearningmedia.org/resource/fin10.socst.personfin.manage.retirement/saving-for-retirement
* “The Benefits of Investing Early” Lesson from EconEdLinks http://www.econedlink.org/lessons/index.php?lid=603&type=educator

**Instructional Tips/Strategies/Suggestions for Teacher**

Instructional notes are embedded in the Lesson description below, indicated by the following symbol: >>>

**Assessment**

* Review of Logarithms (Handout 3) (Optional)
* Investigating the Inverse of Exponential Functions, Part 2 (Handout 4) \*
* Practice Round: Composition with Exponents and Logs
* Composing Exponential and Logarithmic Functions (Handout 5) (Problem #3) \*

\* Items marked with asterisks (\*) above may be used, whole or in part, as formative assessments to gauge students’ understanding and/or skill during group work.

**Lesson Details (including but not limited to:)**

**Lesson Opening**

Doubling Your Earnings (HANDOUT 1)

*We have been exploring the meaning of inverse and inverse functions, using multiple representations, graphical, tabular, and symbolic, as well as verbal representations through the context of real-world financial applications. Now we will explore a particular type of function that should be very familiar to you from prior courses, exponential functions, and apply our learning about inverses to exponential functions.*

Post the problem (from Handout 1) on the board or projector:

*You have been hired as temporary help for the next 30 days. On the first day of your job, your employer asks how you would like to be paid. You are given two options:*

***Option 1:*** *$100 per day*

***Option 2:*** *On day 0, you earn 1 cent. Every subsequent day, you earn double.*

Take a quick survey of the class of who prefers Option 1 vs. Option 2; ask them to respond with an initial reaction/prediction, without doing any calculations. Have students skip to problem 4 (on Handout 1), and give them about 30 – 60 seconds (individually) to write out their initial ideas on an index card. Then have them switch cards with a partner and share their ideas. Ask a few students to share their reasons with the whole class. At this point, just see if they recall this doubling problem from prior courses, and gauge any other ideas they are raising as they initially consider this problem. It does not have to be explained too formally yet; just gather initial ideas.

At this initial stage, see if anyone recognizes what type of function the situation represents (they should recall that this is an exponential function). Ask them if exponential functions have inverses (show of hands for yes or no). Then have partners continue with the rest of the problem.

>>> Teacher Note: Students will be working in cents and dollars. Students may want to use a calculator. It is not necessary, but it may help some students.

>>> Teacher Note: Problem 3 requires students to invert the normal exponential function. Instead of the typical problem setup which asks to find amount for a given time value, the problem asks to find *time for a given amount*. This is the type of question that we foreshadowed early in this unit; after all of the work students have done with inverse functions to this point, they will now be able to model and solve this type of question (SMP4 Model with mathematics). We will use this simple earnings example here, and then return to financial applications. Ask students to recall, and remind them, what an inverse function does — it looks for the input from the output. Have them reason about Problem 3 before solving, both about the solution process and also about the meaning of inverse that they can bring to their thinking about the problem. (SMP2 Reason abstractly and quantitatively)

*Possible solution for Problem 3:*

524288 = 2x

Students will solve by trial and error or may use technology to graph two functions: y1 = 2x and y2 = 524288.

Ask pairs to share how they approached part b of problem 3, how the equation could be solved algebraically (symbolically). Some students may be able to write the equation log 2 524288 = x and evaluate it on a calculator. Others may not make the connection to logarithms, but look for anyone who tries to make a connection between an exponential function and its inverse, as a lead-in to the topic of this lesson. Conclude this opening activity by asking students to revisit their initial predictions, and take a re-vote of the class on which option they prefer. Ask for a few volunteers to share with the class how their thinking changed.

**During the Lesson**

Investigating the Inverse of Exponential Functions, Part 1 (HANDOUT 2)

In this investigation, students will complete a table for the **inverse function** of 2x. Have students work in pairs to complete tables in problems 1 and 2, and remind them to record their strategies as well as their answers, and share their process with their partner.

Students may use trial and error for Problem 2, or they may notice patterns and use some interesting reasoning. Look for the variety of approaches students may be using. Prompt students’ thinking with these questions:

* What does each value mean in the context of the original problem (in Handout 1)? (Encourage students to map back their values to their meaning in the situation, in addition to finding the values in the tables) (MP4)
* What is the difference in the process of finding the inverse of an exponential function vs. other functions? (none…)
* What additional representations might help you (or did you use) to solve these problems, other than tables? How did the additional representation(s) help your thinking (i.e., what additional information did you gain)?
* What was significant about the values you tested to make the value of the inverse function a negative number? Or to make the value of the inverse function 0?

>>> Teacher Note: For guidance and additional scaffolding for students who may be having difficulty with tables or reasoning backwards, use the tips below.

We know that the function f(x) = 2x takes the number 3 (in the context of the problem, day 3) and outputs the number 8 (as the amount earned that day). So how does the inverse function work? (Encourage students to reason backwards, Method 1 from Lesson 4). The inverse f-1(x) must input the number 8 (the amount earned) and output a 3 (the day).

After students complete the table for the inverse of 2x, see how students complete a table for the inverse of 10x.

To investigate the question being asked by the inverse functions above, it may be easier to first discuss the relationship between other familiar inverse functions, such as the square root function , which is the inverse of x2, or the function x/2, which is the inverse of 2x. Ask students to think about the mathematical question one might ask when they see (the inverse of x2). This will help them reason through the process, not only by calculations, but by verbally expressing their thinking to help them make sense of the process of backward reasoning (MP2).

|  |  |
| --- | --- |
| Inverse of x2  (x > 0) | |
| x | y |
| 9 | 3 |
|  |  |

Ask students to think of a question that this function is asking. That is, try to write a question that will reliably take an input value, a number on the left-hand column (in the table above), to an output value, a number in the right-hand column. For example, “What number, squared, gives me x?”

Now have students think about the inverse of the function 2x. Some students might mistake their thinking about 2x in terms of x2; remind them it is very different (recall exponential functions, note the variable as the exponent vs. the base). Similarly, ask students to think of a question that this function is asking. For example, what power of 2 will give me 8? Do the same for the inverse of 10x. For example, what power of 10 will give me 10?

Wrap-up this investigation by asking students to share their strategies as a class, and highlight interesting approaches you observed while visiting groups. Then have students apply their learning by creating a word problem related to compound interest that requires using an inverse to solve it. They don’t have to solve it; they will just use their reasoning to write a problem and connect the parts of the situation to the components of the mathematical model, as in the employer problem earlier. (SMP2 Reason abstractly and quantitatively; SMP4 Model with mathematics)

Review of Logarithms (HANDOUT 3) (OPTIONAL)

This review of logarithms from students’ learning in Algebra 2 is provided as optional practice for students, in preparation for the next part of the investigation. You may also use this as a formative assessment to gauge students’ recall of logarithms.

>>> Teacher Note: In Algebra 2 (standard F.LE.4), students learned to express the solution to an exponential equation, such as 2x = 50, as a logarithm. They did so by writing x = log250 and evaluated the logarithm (using technology, if needed). Students were not required to understand the inverse relationship of these two functions. This lesson applies students’ learning about inverse in this unit to solving exponential equations. In particular, they will draw on the composite principle of functions and their inverses.

Investigating the Inverse of Exponential Functions, Part 2 (HANDOUT 4)

Students will now explore the graphical relationship between f(x) = 2x and its inverse f-1(x), by graphing both functions on the same set of axes. Students should work in pairs or groups of 3 on this investigation. Then ask for a report-out from 4 different groups – each group shares their work on a different problem (there are 4 problems in Handout 4).

Discussion questions for students to consider, while you visit groups and/or during whole class discussion afterward:

* How did you reason about the graphs, without knowing the symbolic representation (equation)?
* What characteristics have we learned about the graphs of inverse functions, in comparison to the original function?
* What do we know about inverse relationships, that is true for all functions and should be true here? (Draw out ideas from about the exponential function and making sense of the inverse of an exponential function; see Teacher Note below).

>>> Teacher Note: This graphical exploration draws on students’ intuition; by this point, they should recognize that in order to graph the inverse, we don’t necessarily need the symbolic representation (equation), because we know the behavior and properties of inverse functions. Students should use their prior learning in this unit to make the graphical representation using reasoning (SMP2 Reason abstractly and quantitatively). Revisit what students have learned from inverse graphs, particularly, symmetry about the line y = x. Lead students to interpret the inverse relationship both in mathematical terms and in the general context of a real-world financial application. Specifically, lead students to the conclusion that this exponential function represents growth of by a certain investment amount for a given rate over time, while the inverse function reverses that statement – it finds the *time* needed for an investment amount to grow at that rate (MP4). This is the component of the problem introduced Lesson 1 that we could not solve previously symbolically; we will revisit that problem at the end of this lesson.

*Now the question is, what do we call this kind of function? As noted on the handout, we use the term logarithm to define the function that finds the time for a given amount earned as f-1 (x) = log2 x.*

As groups are working, ask students to also graph f(x) and f-1(x) on their graphing calculators, and have them analyze the characteristics of the graphs (domain, range, increasing or decreasing behavior, symmetry, asymptotes, end behavior). (SMP2 Reason abstractly and quantitatively) Have them discuss, with their partner, how the two graphs are visually related to each other.

Review this investigation as a whole class by enagaging students in a whole class discussion about Problem 4. Launch into a brief discussion about the concept of logarithms developed in Algebra 2, and how it connects to the concept of inverse functions. Gauge whether students are not only recalling their work with logarithms, but more importantly, beginning to see the following:

* Previously we saw the logarithm as simply a symbolic expression used as a definition to solve an equation in the form abct = d
* Now we can see that the logarithm is actually a function, just like exponential functions are a family of functions
* The logarithmic function is the name of the inverse of an exponential function
* The graph of logarithmic functions are related to the graphs of exponential functions in all the same ways we learned that the graphs of inverse functions are related to their original functions (i.e., symmetry about the line y = x)

Formalize the definition of logarithmic function after the discussion, drawing on students’ comments in your summary.

**For x > 0 and b > 0, and b ≠1, f(x) = logb(x) *if and only* if bf(x) = x**

Ask students to discuss the constraints on x and b. Analyze the notation of the statements in the definition: *where* in the equations the variables are, in relation to being a base or an exponent. Reiterate that “a logarithm is an exponent” (from prior courses).

Pause and Reflect

Ask a challenge/reflection question in preparation for the next part of the lesson (see below). Have students write their ideas on a post-it note and post on flipchart or on the wall; revisit after the next activity (Handout 5) to see how students’ ideas compared to their emerging work and understandings.

* Since the properties of inverse functions apply to all families of functions (for those that have inverses), what principles might we already be able to predict about the relationship between the tabular representations of exponential and logarithmic functions? How about the relationship between the symbolic representations?

>>> Teacher Note: Examples of the kinds of points students should be considering and raising include: (a) Logarithmic functions are inverses of exponential functions; (b) in a table or in consideration of domain and range, the inputs and outputs are switched; (c) 1-to-1 correspondence between domain and range of functions, which also holds true for the inverse if the inverse is a function; (d) the symbolic representation of the composition of an exponential function with a logarithmic function should yield x; and (e) a point on the graph of f(x) shows the amount of money earned on a given day (day, amount), while a point on f-1(x) shows the day that an amount of money is earned (amount, day). Students may not raise all of these ideas, or may raise them in part, which is fine, this is just preparation for the next phase of the lesson. As an instructional strategy, keep a running list of “True statements about Functions and their Inverses” as ideas emerge, and revisit all of them and formalize these principles at the end of the lesson.

Practice Round: Composition with Exponents and Logs

Before beginning the next investigation, have students do a quick activity to practice students’ skill with using composition, this time with examples that are exponential and logarithmic functions. Use examples that are easy to solve, in a variety of different bases and exponents. Have students work in teams of 2. A different example (pair of functions f(x) and g(x)) could be assigned to each team, and also have them make up one of their own, and then each team could switch with another team to verify their work and to try the other team’s example. For example, a function pair could be f(x) = 10x and g(x) = log10x. You can also use this activity as a formative assessment to gauge students’ comfort and fluency with composition (from their learning in Lessons 3 & 4).

Composing Exponential and Logarithmic Functions (HANDOUT 5)

Students will now work in pairs on Handout 5, to explore the principles of inverse functions as they apply to the symbolic representations (equations) of exponential functions and logarithmic functions, in particular. Revisit the statements students generated in the “Pause and Reflect” discussion (above) as a lead-in for students to keep in mind as they do this investigation.

>>> Teacher Note: Be sure to emphasize that students record not only their solutions, but especially their observations and reasoning as they explore the problems. These are leading to a generalization about composition of exponential and logarithmic functions, that will apply what students have already learned (in this unit) about composition with any inverse functions. (MP8)

Tips and questions for students to consider as they work toward the generalization in problems 2 and 3:

* How do you interpret, in words (verbally) and in general notation, the meaning of f(g(x)) and the g(f(x)) for exponential and logarithmic functions?
* How could you re-write the statements using f and f-1? Why?
* Are the statements f(g(x)) = x and g(f(x)) = x always true for exponential and logarithmic functions? How do we know? What else do we know about restrictions on the domains?
* Explain the meaning of the generalization, in your own words (shown on Handout 5):

**Finding the Exponent of a Logarithmic Function by Composition**

Since f(g(x)) = x, then

**Finding the Logarithm of an Exponential Function by Composition**

Since g(f(x)) = x, then

>>> Teacher Note: For students who need help, or for the whole class as a guidance before students continue with further practice, remind students that to create and solve an equation of their choice using the inverse properties above, they will need to use their emerging understanding of the inverse of an exponential function (a logarithmic function) to rewrite the equation. Ask students how they handled the decomposition of functions (below) earlier in this unit (they broke up 1 function in order to see it as the composition, which actually generates a new function). See if students recognize the need for a third function. For visual learners or as needed with all students, refer back to the pictorial representation using lemons, apples, and oranges. Have students label these functions in terms of the fruits and make a new picture to represent this example, if it’s helpful.

You can use the following example to model or offer guided instruction:

For the function 2x+1 = 3, ask students to think of the each side of the equation as a separate function (recall a similar decomposition strategy in Lesson 3). 2x+1 = 3 can be considered 2x+1 = f(x) and 3 = g(x). Students should recognize that the variable for which they need to solve is actually in the exponent of f(x).

In this example, introduce a third function h(x) = log2x (since we are working with an exponential function) and compose it with each function f and g:

log2 2 x+1 = log2 (3) h(f(x)) = h(g(x))

x + 1 = log2 3 log2 2 x+1 = x + 1 by the composition of inverse functions properties f(f-1(x)) = x and f-1(f(x)) = x

x = 1 – log2 3

Discuss the interpretation of the solution. Ask students to recall that log2 3 is a numerical value that can be calculated.

Discuss why we introduced h(x) = log2x as a function with a base of 2, instead of a different base.

# Finding the Inverse of a Logarithmic Function by Composition

Gather ideas from students on how they might find the inverse of a logarithmic function, now that they know the logarithmic function is the inverse of an exponential function, so how would they go backwards? Then lead students in a mini-lecture (as students do the problem and answer questions) to model the logic and reasoning for finding the inverse of a logarithmic function, and ask them to draw on their understanding of composition and verification by composition to make sense of this process.

*If the logarithmic function is one-to-one, its inverse exits. The inverse of a logarithmic function is an exponential function. When you graph both the logarithmic function and its inverse, and you also graph the line y = x, what should you see in the graphs of the logarithmic function and the exponential function?* *They are mirror images of one another with respect to the line y = x. In other words, a logarithmic function and its inverse are symmetrical with respect to the line y = x.*

Find the inverse of **f(x) = log (x + 1)**

*Recall that the composition of a function with its inverse will take you back to where you started:* **f(f-1(x)) = x**

*Try this. Rewrite this principle for inverse functions in terms of f(x) in the example (above).*

*So* f(f-1(x)) = x *in terms of this function means* f(f-1(x)) = f(log (f-1(x) + 1)

*Recall that the base of the log function is 10 (also called the common log), the exponent is x, and we can use the definition to convert the problem to the exponential function:*

10x = f-1(x) + 1

*We can solve this equation – not for x, as you have been doing for many years, but what would we solve for, as we saw in Lesson 4?*

*We can solve for f-1(x) – the inverse function.*

f-1(x) = 10x – 1

Finding Domain and Range of the Inverse Function

*Recall that the domain of f(x) is equal to the range of f-1(x). The range of f(x) is equal to the domain of f-1(x).*

*The domain of f(x) is (-1, ∞) and the range of f-1(x) is also (-1, ∞).*

*In a graphical representation, this means that the entire graph of f(x) will be located to the right of the vertical line x = - 1, and the entire graph of f-1(x) will be located above the line y = - 1.*

Verify the Results

*Check by finding points on both graphs. In the original graph, f(99) = log (99 + 1) = 2. This means that the point (99, 2) is located on the graph of f(x). If we can show that the point (2, 99) is located on the inverse, we have shown that our answer is correct, at least for these two points.*

*Since f(2) = 102 – 1 = 99, the point (2, 99) is located on the graph of the inverse function. We have correctly calculated the inverse of the logarithmic function f(x). This is not the “pure” proof; however, it works at an elementary level.*

Practice

Now give students a few basic logarithmic functions as additional examples, to practice finding the expression for the inverse function (which will be an exponential function), the domain and range, and verify their results as we did above.

Problem 3 (in Handout 5)

Use the last problem in the investigation as a creative problem-solving opportunity and, also, a formative assessment. Teams of 2 create an equation that can be solved using either of the properties we have explored – using composition to find the exponential from a logarithmic function or finding the logarithmic from the exponential function. Have teams switch their equations with another team and find the solution. The “review teams” should find the inverse function using composition, find the domain and range, and verify their results as we did above, and also graph the original and inverse functions. Then teams switch back, and “original teams” review the work to make sure it meets their original intent.

**Lesson Closing**

Million Dollar Problem, Revisited

Return to the graph of the problem students initially explored in this unit (Lesson 1 Handout 3: Investigation: Part 1, Opener). Ask students to recall their work and their analysis at the time (they only used graphical and tabular representations), and what value they were not able to find symbolically (time). Revisit/discuss the meaning of the context that the graph models. Then have students, individually first, explore these questions:

1. *Verify the meaning and the value of the point P, using the composition method.*
2. *Find the number of years it would take to reach $1 million based on that function A(t), using a symbolic approach.*

*First, explain how you would set up the problem and what a symbolic approach to analyzing and solving this mathematical model means in the context of the problem. Then work on the solutions.*

>>> Teacher Note: This problem can be used as a formative assessment and/or further reviewed by a partner after students have thought about and worked on the problem alone. This is an opportunity for students to apply their learning in a modeling context and, even if they do not reach the solution, they should be able to demonstrate reasoning through thinking about the situation based on the principles of inverse functions they have learned in this unit. (MP2, MP4)

More Financial Applications

In Lesson 2, students learned a bit about financial applications in the real world and the role of a financial planner. As another hook, and further preparation for the CEPA, have students explore another financial application, this time in the context of retirement (this could also be assigned as homework). Watch the “Saving for Retirement” PBS Video and engage students in “The Benefits of Investing Early” lesson (see Instructional Resources section).

>>> Teacher Note: The suggested video offers another reminder on the role of a financial planner. The recommended lesson offers online calculators and does not address financial applications to the level of depth in this unit. The lesson is simply provided as an opportunity for students to gain context about retirement considerations, since it will be relevant for the CEPA.

**Lesson 5 Handout 1**

**Doubling Your Earnings**

You have been hired as temporary help for the next 30 days. On the first day of your job, your employer asks how you would like to be paid. You are given two options:

**Option 1:** $100 per day

**Option 2:** On day 0, you earn 1 cent. Every subsequent day, you earn double.

Answer the following questions based on the problem above.

1. How many pennies do you have on day 10? Explain.

2. How many pennies do you have on day n? Explain.

3. On *which day* do you have 524,288 cents?

a. Before you answer, express this question as an equation in terms of x.

b. Now, find x. How did you solve it? How would you solve it algebraically (symbolically)? Explain your reasoning.

4. Which option would you prefer that your employer pays you? Why?

**Lesson 5 Handout 2**

**Investigating the Inverse of Exponential Functions, Part 1**

1. We know that the function f(x) = 2x will take the number 3 as an input, and give 8 as an output. Complete the table below for your choice of values of x.

|  |  |
| --- | --- |
| **x** | **2x** |
| 3 | 8 |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

2. We also know that the inverse function f-1(x) *must* take the input 8 and give an output of 3.

a. Fill in the table for the **inverse function of 2x**.

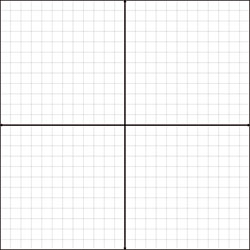
|  |  |
| --- | --- |
| Inverse of 2x | |
| ***x*** | ***y*** |
| 8 | 3 |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |
|  |  |

b. Which x-values would give the inverse function a value of 0? How about a negative number? Can you predict without calculating? Explain your reasoning.

**Lesson 5 Handout 3**

**Investigating the Inverse of Exponential Functions, Part 2**

1. Graph f(x) = 2x and its inverse on the same coordinate plane, below.



2. What are the domain and range of f(x)? What are the domain and range of f-1(x)?

3. Revisit the Doubling Your Earnings exploration (Lesson 5 Handout 1). Describe what x and 2x represent in the context of the problem.

## Formal Definition of Logarithmic Function as the INVERSE of bx:

f(x) = logb(x) *if and only if* bf(x) = x, for x > 0, and b > 0 , and b ≠ 1

4. Based on this definition, what is the *equation* for the *inverse* of f(x) = **2x**? Explain.

**Lesson 5 Handout 4**

**Review of Logarithms**

The expression log28 can be described in words as asking the question, “2 to what power is 8?”

Describe the following logarithmic expressions in words.

1. log28

2. log22

3. log1010

4. log10100

5. logee

6. loge1

7. Revisit your descriptions in problems 1-6. Make a generalization: what does the expression logbx tell you about x?

8. Yasmin invested $200 in a savings plan to double her money annually. After y years, the amount of money Yasmin will have in the bank can be expressed as (200)(2x) dollars. Today, Yasmin’s savings account has $1000. How many years have passed? Express your answer in terms of a logarithm.

**Logarithms Review**

**If x > 0, then we define log10x as a number y that satisfies the equation 10y = x.**

**The expression log10x is commonly written as log x. We call log x the common logarithm of x.**

**y = log x** if and only if **10y = x**

Example 1 Example 2

log10100 = log 100 = 2 log100.01 = log 0.01 = −2

because 102 = 100 because 10−2 = 0.01

**The expression logex is commonly written as ln x. We call ln x the natural logarithm of x.**

**y = ln x** if and only if **ey = x**

**Lesson 5 Handout 5**

**Composing Exponential and Logarithmic Functions**

**1. Let f(x) = 2x and g(x) = log2x**

a. Find f(g(8)) b. Find g(f(8))

c. Describe everything you know about the functions f(x) and g(x) (including their graphs).

**2. Suppose f(x) = ax and g(x) = logax**

a. Find f(g(x)) b. Find g(f(x))

**Finding the Exponent of a Logarithmic Function by Composition**

Since f(g(x)) = x, then

**Finding the Logarithm of an Exponential Function by Composition**

Since g(f(x)) = x, then

**3.** Solve an equation using the properties above.

# Curriculum Embedded Performance Assessment (CEPA)

**Planning for a Solid Financial Future**

**Standard(s)/Unit Goal(s) to be addressed in this CEPA:**

**FRDM.4** Make financial decisions by systematically considering alternatives and consequences

**SI.3** Evaluate investment alternatives.

**F-BF.B.4b** (+) Verify by composition that one function is the inverse of another.

**F-BF.B.4c** (+)Read values of an inverse function from a graph or a table, given that the function has an inverse.

**F-BF.B.5** (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

**SMP.3** Construct viable arguments and critique the reasoning of others.

**SMP.4** Model with mathematics.

**RCA-ST.CS.4** Determine the meaning of general academic vocabulary as well as symbols, key terms, and other domain-specific words and phrases as they are used in a specific or technical context relevant to grades 11-12 texts and topics.

This final performance task was designed to bring the work on inverse functions, including exponential and logarithmic functions, into an authentic financial literacy context. The task measures students’ understanding of financial decision-making combined with their use of graphical, tabular, and symbolic representations of inverse functions to model and solve financial problems. Students take on the role of a financial planner to evaluate and recommend the risks and advantages of a series of financial investment options (SI.3) and make financial decisions by systematically considering alternatives and consequences (FRDM.4). To provide the best advice for their clients, they will need to develop, analyze, and verify exponential models for a variety of investment scenarios (F-BF.4b, F-BF.4c, SMP.4), apply the inverse relationship between exponential and logarithmic functions to solve problems (F-BF.5), and propose viable mathematical arguments to justify their reasoning (SMP.3). The task also involves considering financial information in a technical context (RST.CS.4) and communicating mathematical claims professionally within an authentic task.

**CEPA Teacher Instructions**

**Planning for a Solid Financial Future**

1. Remind students that they have previously read the “Save and Invest” article on investment and creation of wealth, and viewed the PBS videos on savings and planning for retirement. The article, especially, provides a solid background for you to reference as you attempt to give sound financial advice to your clients.

2. Review and discuss the goal, role, and audience of the task (CEPA Handout 1). Students should have the opportunity to ask questions about the task before starting, to clarify any of the language or description.

3. Review and discuss the Criteria for Success (CEPA Handout 1) and rubric (CEPA Handout 9) with the class, to orient them to the expectations for defining a strong mathematical response. Students should have the opportunity to ask questions about the criteria and the rubric before starting the task, to clarify any of the language or description.

>>> Teacher Note: Reiterate that the graphs of the exponential function and the inverse, as well as the series of calculations and graphical representations, are products in and of themselves, but that these particular products should be cited and incorporated as evidence for the recommendation to the client.

4. Review the list of financial investment options available to students in their role as financial planner, to give them the opportunity to ask any questions they may have.

>>> Teacher Note: If students do not ask, clarify the meaning of “minimum” contribution. If students ask questions about specific investment vehicles (options), refer them back to the “Save and Invest” article, which outlines different options for saving and investing.

5. Provide supporting resources and handouts (CEPA Handouts 1-9). These include templates for email responses, but responses certainly do not need to be limited to only the space provided.

>>> Teacher Note: Students should work individually with the expectations and rubric in hand to craft a suitable response to each of the emails. These can be done sequentially or handed out all together. If technology is available to students, they can type email responses in Word and generate graphs using technology.

**CEPA Handout 1**

**Student Instructions**

**Planning for a Solid Financial Future**

You are a financial planner (Louis Barajas\*). You have three clients with different careers, lifestyles, and expenses (Eddie Romero\*, Connie Tuckman, and Lashonda Jackson). Each client has sent you an email describing their particular concerns, along with a little background about their lifestyles and financial interests. Your goal is to provide the best possible financial plan to address your clients’ financial concerns and help them make sound financial decisions. Your company offers five primary financial investment and savings options; you will evaluate the alternatives and respond to your clients’ emails with recommendations. In your emails, you will provide clear evidence to support your advice, including mathematical models for the different situations, graphical and symbolic representations of the inverse functions that model each client’s situation, and professional communication that provides a mathematical justification for your recommendation.

## Criteria for Success

1. Email Response

Develop an email response for each client (use the templates provided).

Each email (1 for each client) should include:

* A clear and concise description of your recommendation to the client with a proposal for at least one of the five available investment options
* A summary of the alternatives you considered, your reason(s) for choosing the options you are proposing, and how you think your recommendation fits with your client’s lifestyle and financial needs
* For each option you are recommending, provide at least 1 benefit and 1 risk involved

2. Mathematical Evidence

For each client, develop a collection of evidence that supports your recommendation.

Your evidence (a set for each client) should include:

* The mathematical function(s) that model the situation, with an explanation of your reasoning involved in determining the function(s)
* Mathematical representation of the investment option(s) that you considered relevant to the situation – include the graphical and symbolic (equation) representations of the functions and their inverses
* Verification of the inverse functions that model the problem, using composition
* An accurate and logical sequence of calculations and reasoning involved in analyzing and solving the problem for each client’s specific context
* A concise, persuasive chain of reasoning connecting your recommendation to the evidence you have provided

\* From the PBS video you watched in Lesson 2. Only the characters’ names are used in this CEPA. However, none of the financial information from the video is relevant for this task.

**CEPA Handout 2**

**Financial Investment Options Available**

|  |  |  |
| --- | --- | --- |
| **Regular Savings Account** | **Annual Interest Rate** | **Minimum Contribution to Open an Account** |
| Savings Account | .07% | $0 |
| Money Market Savings Account | 1.3% | $500 |
| Low-Risk Mutual Fund | 6.3% | $5000 |
| High-Risk Mutual Fund | 8.3% | $7000 |
| Individual Retirement Account (IRA) | 6.15% | $5000 |

**CEPA Handout 3**

**Client 1 (Eddie Romero)**

|  |  |
| --- | --- |
| To: | louis.barajas@financialplanmaker.com |
| From: | eddie.romero@bestemail.com |
| Subject: | Budget and bonus questions |

|  |
| --- |
| Dear Louis,  It was great to meet with you last week. As soon as I left our meeting, I begin to think more and more about saving and being more prepared for some big purchases coming down the road.  I just heard that I am receiving a bonus at work and thought I could use that to start saving for a car. My bonus turned out to be $1200. I would like to know what type of return on this investment I could expect in 5 years, so I can have a good down payment for a new car.  Thanks for the help,  Eddie |

**CEPA Handout 4**

**Response to Client 1 (Eddie Romero)**

|  |  |
| --- | --- |
| To: | eddie.romero@bestemail.com |
| From: | louis.barajas@financialplanmaker.com |
| Subject: | Re: Budget and bonus questions |

|  |
| --- |
|  |

**CEPA Handout 5**

**Client 2 (Connie Tuckman)**

|  |  |
| --- | --- |
| To: | louis.barajas@financialplanmaker.com |
| From: | connie.tuckman@bestemail.com |
| Subject: | Moving to the city |

|  |
| --- |
| Hi Louis,  When we last talked I was renting in the suburbs, but I have recently moved into the city and will be selling my car. I am hoping to get about $12,000 for my car and want to invest this money both for retirement, and also as an emergency fund.  As you know, I am turning 45 this year and want to put more away, so I can truly retire when I’m 65. I would like to add another $20,000 to my existing retirement funds from this investment. Can you help determine how to invest, and how much of the $12,000 should go to the retirement funds?  As always, thank you for your help.  Connie |

**CEPA Handout 6**

**Response to Client 2 (Connie Tuckman)**

|  |  |
| --- | --- |
| To: | connie.tuckman@bestemail.com |
| From: | louis.barajas@financialplanmaker.com |
| Subject: | RE: Moving to the city |

|  |
| --- |
|  |

**CEPA Handout 7**

**Client 3 (Lashonda Jackson)**

|  |  |
| --- | --- |
| To: | louis.barajas@financialplanmaker.com |
| From: | lashonda.jackson@bestemail.com |
| Subject: | Change of Plans |

|  |
| --- |
| Hello Louis,  We haven’t talked since we made a financial plan together over a year ago, but I would like to make some changes based on an exciting family event. I have a new baby niece, Kendra!  Kendra is wonderful and I would really like to set aside some money for her college education. I know that UMass costs about $18,000 a year now, and I would like to pay for her first year of college 18 years from now. It looks like tuition is going up about 1.25% every year and I want to be prepared!  How much do I need to put away now so that I can give this gift to my niece? Could you provide me a couple different investment scenarios?  Thanks so much,  Lashonda |

**CEPA Handout 8**

**Response to Client 3 (Lashonda Jackson)**

|  |  |
| --- | --- |
| To: | lashonda.jackson@bestemail.com |
| From: | louis.barajas@financialplanmaker.com |
| Subject: | Re: Change of Plans |

|  |
| --- |
|  |

**CEPA Handout 9**

**Rubric**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | **Exemplary** | **Proficient** | **Developing** | **Emerging** |
| **Graphs**  *Key Question*:  Did you provide your client with clear, accurate, and user friendly graphical representations of your recommendations? | 1. Your graph is completed titled and your axes are labeled including units of measurement.  2. Your graph is carefully and accurately created with precision indicating a clear sense that the audience is an actual client.  3. Your graph is drawn and labeled over the relevant intervals and clearly communicating the particular solution asked for by the client | 1. Your graph is accurately and clearly labeled with a title and one axis or both axis are labeled with the correct units.  2. Your graph is correctly constructed with some precision  3. Your graph is drawn over the correct intervals that are relevant to the question and task from the client. | 1. Your graph is titled, but the axes are not labeled or labeled without the correct values.  2. Your data points or curve is somewhat accurate, but includes incorrect points or poor sketching of the curve  3. Your graph shows the graph over an interval, but the interval is not relevant to the presentation of the information to your client. | 1. Your graph did not use any labels to convey the meaning of the values  2. Your data points are not accurately plotted.  3. Your graph does not include the appropriate domain and range intervals |
| **Equations**  *Key Question:*  Did you show and provide algebraic evidence of your recommendations in a logical and step-by-step approach? | 1. Your variables clearly and accurately defined.  2. Your equation or function is accurate and complete and the appropriate domain and range given for its use. | 1. Your variables were accurately defined.  2. Your equation or function is an accurate and complete model of the situation | 1. You variables were defined, but vaguely or inaccurately.  2. You wrote an equation or function with a single error in a equation parameter. | 1. Your variables were undefined.  2.You wrote an incomplete or inaccurate equation or function to model the situation |
| **Strategies and**  **Reasoning**  *Key Question*:  Is there evidence that  you proceeded from a  plan, applied  appropriate  strategies, and  followed a logical  and verifiable  process toward a  solution? | 1. You chose innovative and  insightful strategies for solving  the problem.  2. You proved that your solution  was correct and that your  approach was valid.  3. You provided examples and/or counterexamples to support your solution.  4. You used a sophisticated  approach to solve the problem.  5.  You provided multiple pieces of evidence to support your solution. | 1. You chose appropriate,  efficient strategies for solving  the problem.  2. You justified each step of your work.  3. Your representation(s) fit the task.  4. The logic of your solution was apparent.  5. Your process would lead to a complete, correct solution of the problem.  6.  The evidence you showed  clearly supported your solution. | 1. You used an oversimplified approach to the problem.  2. You offered little or no explanation of your strategies.  3. Some of your representations accurately depicted aspects of the problem.  4. You sometimes made leaps in your logic that were hard to follow.  5. Your process led to a partially complete solution.  6.  The evidence for your solution was inconsistent or unclear. | 1. Your strategies were not  appropriate for the problem.  2. You didn’t seem to know where to begin.  3. Your reasoning did not support your work.  4. There was no apparent  relationship between your  representations and the task  5. There was no apparent logic to your solution.  6. Your approach to the problem would not lead to a correct solution.  7.  You gave no evidence of how you arrived at your answer. |
| **Overall Communication**  *Key Question*:  Was your client able to easily understand your thinking without making inferences and guesses about your recommendation? | 1. Your explanation was clear and concise.  2. You communicated concepts with precision.  3. Your graphs and equations expanded on your solution.  4. You gave an in-depth  explanation of your reasoning.  5. You used mathematical  terminology precisely. | 1. Your solution was clear and the client could understand what you did and why you did it.  2. Your solution was well  organized and easy to follow.  3. Your solution flowed logically from one step to the next.  4. Your graphs and equations helped clarify your solution.  5. You used mathematical  terminology correctly. | 1. Your solution was hard to follow in places.  2. Your explanation required your client to make inferences about what you meant in places.  3. Your explanation was redundant in places.  5. Your graphs and equations were somewhat helpful in clarifying your thinking.  6. You used mathematical terminology imprecisely. | 1. Your solution was difficult to follow.  2. Your explanation seemed to ramble.  3. You gave no explanation for your work.  4. You did not seem to have a sense of what your client needed to know.  5. Your graphs and equations did not help clarify your thinking.  6. You used mathematical  terminology incorrectly. |