Contents

Major Theme: Options for course-taking sequences, including offering the high school model course algebra I in grade 8 and pathways to advanced courses, such as calculus by the end of grade 12

Why this issue is important:

The Massachusetts Curriculum Framework for Mathematics represents an opportunity for districts to revisit and plan course sequences in middle and high school mathematics along with educators, middle and high school guidance counselors, stakeholders, including parents, and mathematics leaders. To help inform these decisions, the following information should be considered:

- The rigor of both the grade 8 and High School Model Course Algebra I standards;
- The offering of the High School Model Algebra I course standards in middle school to students for whom these standards are appropriate; and
- Options for high school pathways that accelerate learning starting in grade 9 to allow students to reach advanced mathematics courses such as Calculus by grade 12.

The pre-kindergarten to grade 8 mathematics standards present a coherent progression of concepts and skills that will prepare students for the High School Model Algebra I course. The grade 8 standards are rigorous; students are expected to learn about linear relationships and equations and begin the study of functions and compare rational and irrational numbers. They also learn about two and three dimensional space and figures and apply the Pythagorean Theorem in problem solving. In addition, the statistics presented in the 2011 grade 8 standards are more sophisticated than those in our previous framework. Students will need to progress through the grade 8 mathematics standards in order to be prepared for the Model High School Algebra I course and standards.

Some students may master the standards designated for grade 8 in earlier grades. This would enable these students to take the High School Model Algebra I course in middle school. For those students ready to move at a more accelerated pace, one method is to compress the standards for any three consecutive grades into an accelerated 2-year pathway. The selection and placement of students into accelerated opportunities must be done carefully in order to ensure student success.

For students who study 8th grade standards in grade 8, there are a variety of accelerated pathways that will lead them to advanced mathematics courses through the end of grade 12, such as Calculus. Some pathways will include advanced courses such as: the Model Precalculus course, the Model Advanced Quantitative Reasoning course, or courses not included in the Framework, such as: Statistics, Discrete Mathematics, Linear Algebra, AP Statistics, or AP Calculus.

Recommendation:
- **Update the Department's 2011 Guidance Document** titled: *Making Decisions about Course Sequences and the High School Model Algebra I Course.* This document presents information and resources to ground discussions and decision-making about students' course sequences. Including this guidance in the Framework will make it more accessible to educators and the education community.
  - Include the guidance document in the High School section of the revised Mathematics Framework.
  - Revise the graphic of Algebra I and High School Course Pathways to include additional pathways for students pursuing varied careers and college paths. We are revising the graphic in this document that currently shows three pathways that all lead to calculus in grade 12. **The first pathway** compacts the standards for grade 6, grade 7, and part of grade 8 so that students can take grade 8 standards related to algebra and the Algebra I High School Model course standards in grade 8. **The second pathway** doubles up courses in high school; students in grade 9 take Algebra I and Geometry, then in grade 10 take Algebra II (and Geometry if they did not take Geometry in grade 9). **The third pathway** is an enhanced pathway in high school; students enter grade 9 and take enhanced Algebra I in grade 9, enhanced Geometry in grade 10, enhanced Algebra II in grade 11 and Calculus in grade 12. (Precalculus standards are folded into the enhanced courses). We are revising this graphic to respond to students' diverse interests and career and college plans and **will add additional pathways that will lead to other advanced courses**, such as Statistics and Advanced Quantitative Reasoning.

Examples of proposed revisions to standards

To increase coherence

Proposed Revisions to Standards Related to Patterns in Counting

Existing:	Proposed:
Kindergarten Counting and Cardinality Standard 1 K.CC.1. Count to 100 by ones and by tens.	**Kindergarten Counting and Cardinality Standard 1** K.CC.1. Count to 100 by ones and by tens. Recognize the "one more" and "ten more" pattern of counting
Grade 2 Operations and Algebraic Thinking Standard 3 2.OA.3. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g. by pairing objects or counting them by twos, write an equation to express an even number as a sum of two equal addends.	**Grade 2 Operations and Algebraic Thinking Standard 3** 2.OA.3 Identify patterns in odd and even numbers using concrete models or drawings. Determine whether a group of objects (up to 20) has an odd or even number of members, e.g. by pairing objects or counting them by twos, write an equation to express an even number as a sum of two equal addends.
Grade 2 Number and Operations in Base 10 Standard 2 2.NBT.2. Count within 1000; skip count by fives, tens, and hundreds.	**Grade 2 Number and Operations in Base 10 Standard 2** 2.NBT.2. Identify patterns in skip counting. Count within 1000; skip count by fives, tens, and hundreds.

Rationale: Recognizing patterns in numbers is key to mathematics, including the practices of looking for regularity and looking for patterns or structure in our number system. Looking for and finding patterns is foundational for algebraic thinking. These edits explicitly bring this learning progression to younger children while maintaining the critical focus on arithmetic in these early grades.

Proposed New Glossary Entries	**Standards Related to the Fluency Progression in Grades K–6 (emphasis added for reference):**
Fluency: Fluency is knowing how a number can be composed and decomposed and using that information to be flexible and efficient in solving problems. The best way to develop fluency is to develop number sense and to work with numbers in different ways using a variety of strategies. (Parish., S. 2014. Number Talks: Helping Children Build Mental Math and Computational Strategies, Grades K–5. Updated with Common Core Connections, Math Solutions.)	• Kindergarten: K.OA.5. *Fluently* add and subtract within 5. • Grade 1: 1.OA.6. Add and subtract within 20, demonstrating *fluency* for addition and subtraction within 10. Use mental strategies such as counting on; making ten (e.g., 8+6=8+2+4=10 +4=14);

Know from Memory: As students work on meaningful number activities they will commit facts to heart (recall and easily retrieve to use when needed) at the same time as understanding numbers and math. Students are expected to build understanding and then they are expected to know the math facts from memory (Jo Boaler, 2015, Fluency Without Fear. Youcubed.org).

Standard Algorithm: A finite set of efficient steps for completing a procedure based on place value and properties of operations. The standard algorithm for each operation relies on:

- Decomposing numbers written in base-ten notation into base-ten units and then carrying out single-digit computations with those units using the place values to direct the place value of the resulting number, and
- Using the one to ten uniformity of the base ten structure of the number system to generalize to larger whole numbers and to decimals.

The grades 4–6 standards expect students to know how to use the standard algorithms to perform operations with numbers. For example:

$$\begin{array}{r} \mathbf{1} \\ 34 \\ +27 \\ \hline 61 \end{array}$$

Algorithm: An algorithm is a finite set of steps for completing a procedure, e.g., long division. In grade 3 students may use a range of algorithms based on place value and properties of operations. For example:

$$\begin{array}{r} 34 \\ +\ 27 \\ \hline 11 \text{ (sum of ones)} \\ +\ 50 \text{ (sum of tens)} \\ \hline 61 \text{ (final sum)} \end{array}$$

decomposing a number leading to a ten; using the relationship between addition and subtraction; and creating equivalent but easier or known sums).

- Grade 2: 2.OA.2 *Fluently* add and subtract within 20 using mental strategies. By the end of grade 2, *know from memory* all sums of two-digit numbers.
  - MA.2.a. By the end of grade 2, *know from memory* related subtraction facts of sums of two one-digit numbers.
- Grade 3: 3.OA.7. *Fluently* multiply and divide within 100, using strategies such as the relationship between multiplication and division (e.g. knowing that 8x5=40, one knows 40 divided by 5=8) or properties of operations. By the end of grade 3, *know from memory* all products of two one-digit numbers. 3.NBT.2. *Fluently* add and subtract within 1000 using strategies and *algorithms* based on place value, properties of operations, and/or the relationship between addition and subtraction.
- Grade 4: 4.NBT.4. *Fluently* add and subtract multi-digit whole numbers *using the standard algorithm.*
- Grade 5: 5NBT.5. *Fluently* multiply multi-digit whole numbers *using the standard algorithm.*
- Grade 6: 6.NS.2. *Fluently* divide multi-digit numbers using the *standard algorithm*. 6.NS.3. *Fluently* add, subtract, multiply and divide multi-digit decimals *using the standard algorithm* for each operation.

The 2011 Framework lacks definitions for "fluency" "know from memory," "standard algorithm," and "algorithm."

Rationale: There is a very strong and coherent progression that develops students' fluency in number sense and operations. This progression begins in prior grades where they develop an understanding of the concept (for example: addition and subtraction) by exploring number relationships and place value, including grouping in tens and ones. They use models and solve problems. They build their understanding of the relationship between addition and subtraction. Then in grade 2 they become fluent in adding and subtracting within 20 and are able to use a variety of strategies to solve problems; and by the end of grade 2 they are expected to know the addition and related subtraction facts from memory. This same progression of developing understanding and learning is presented for the grade 3 standard that expects students to first fluently multiply and divide within 100, using strategies or properties of operations. By the end of grade 3 students are expected to know from memory multiplication and related division math facts.

To further explain this progression and the key terms used in this progression, we will define these key terms, include them in the glossary, and provide explanatory text with links to related articles, and examples. Note: In our research on other states' revisions, we noted that almost all of these states recognized the need to address the definition of fluency and know from memory. Several states debated whether students should be expected to know from memory the math facts. We propose retaining the two standards that include this expectation and will include the important message that it is critical to develop understanding as a foundation first, but that students need to be able to recall the math facts when needed in order to efficiently calculate and solve more complex calculations and problems as they advance in their mathematical studies.

Examples of proposed revisions to standards	
To increase focus	
Existing Grade 6 Ratio and Proportional Relationships Cluster Heading: Understand ratio concepts and use ratio reasoning to solve problems.	**Proposed: Grade 6 Ratio and Proportional Relationships Cluster Heading:** Understand ratio and rate concepts and use ratio and rate reasoning to solve problems.
Rationale: Grade 6 focuses on ratio, rate, and proportional reasoning. Connecting ratio and rate is an identified critical area for grade 6. The term "rate" was added to explicitly insure that the focus is on both ratio and rate concepts for the standards under this heading.	
Existing High School Model Course Headings: Algebra I – Building Functions: Build a function with integer domains that models a relationship between two quantities. Algebra I – Interpreting Functions: Interpret functions that arise in applications in terms of the context.	**Proposed: High School Model Course Headings:** Algebra I – Building Functions: Build a function with integer domains that models a relationship between two quantities using linear, quadratic, and exponential functions with integer exponents. Algebra I – Interpreting Functions: Interpret functions that arise in applications in terms of the context, functions include linear, quadratic, and exponential functions with integer exponents.
Rationale: Edits were made to course headings and standards to identify **the types of functions** that are the focus of the Model High School Algebra I course standards (this was also done for the Algebra II Model course).	

Examples of proposed revisions to standards	
To increase or maintain rigor	
Proposed Decision: Maintain all (+) Plus High School Conceptual Category Standards.	**Discussion:** Review Panel discussions focused on whether to maintain, move, or edit the (+) Plus High School Standards. The decision was made to keep the (+) plus standards in the Model HS Courses
Rationale: We discussed whether to maintain, edit, or revise the (+) plus standards located in the high school model and advanced high school model courses. In 2011 we were not able to edit any of these broad conceptual category (+) standards and they were included in the model high school courses and advanced model courses where appropriate. The rationale for keeping all of the (+) standards is that, as presented, they are optional and their inclusion provides options for differentiation and further challenge, rigor, and extended learning opportunities for students who are ready for more advanced content related to the course. Any footnotes helping to clarify the content have been incorporated into the wording of the standard itself.	

Examples of proposed revisions to standards

To increase clarity	
Existing: **Grade 4 Number and Operations in Base Ten Standard 2** 4.NBT.2. Read and write multi-digit whole numbers using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers based on meanings of the digits in each place, using <, =, and > symbols to record the results of the comparisons.	**Proposed:** **Grade 4 Number and Operations in Base Ten Standard 2** 4.NBT.2. Read and write multi-digit whole numbers to the millions place, using base-ten numerals, number names, and expanded form. Compare two multi-digit numbers within a million, based on meanings of the digits in each place, using <, =, and > symbols to record the results of the comparisons.
Rationale: To clarify the term multi-digit whole number. The standard was edited to identify the place value (millions place) and boundary (up to and including 1,000,000).	
Existing Standard and Related Footnote: **Model Algebra I, Reasoning with Equations and Inequalities Standard 10** A.REI.10 Understand that the graph of an equation in two variables is the set of all the solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation. *Related footnote: In Algebra I, functions are limited to linear, absolute value, and exponential functions for this cluster.*	**Proposed:** **Model Algebra I, Reasoning with Equations and Inequalities Standard 10** Understand that the graph of an equation (equations to include linear, absolute value, and exponential) in two variables is the set of all the solutions plotted in the coordinate plane, often forming a curve (which could be a line). Show that any point on the graph of an equation in two variables is a solution to the equation. *Footnote eliminated.*
Rationale: In 2011 MA was not able to edit the existing CCSS. In order to develop the Model High School Courses, we had to use the broad Conceptual Category standards to create the standards for the model courses and because of their breadth they often crossed a number of traditional course boundaries. In order to define the content of each course and to present a coherent progression of concepts and skills, we listed the conceptual category standards relevant for each course and then had to create a set of footnotes to explain and clarify the content expectations and boundaries appropriate for each course. The proposed revisions have edited these broader standards, eliminated most of the footnotes and now provide clearer wording and expectations and more focused progressions of standards for each high school course.	

Revisions to Introductory Section and Appendices
Introductory Section
• Shorten introductory section and update it to focus on standards development in Massachusetts 1993-2017.
Standards for Mathematical Practice
• The descriptive texts of the practices were revised by grade span- PK-5; 6-8; and high school. Further feedback suggested that we present even more customized descriptions at narrower grade-spans, such as: PK-1; 2-3; 4-5; 6-8; HS. We are presently developing these descriptions and designing a new format to present more specific examples in a more condensed chart form in an effort to make the standards for mathematical practice more relevant and understandable for teachers of math at all grade levels.
Appendices
• Update Glossary entries to include important terms in the Framework.   • In a digital version of the Framework, hyperlink standards, examples, and glossary terms where appropriate.