# Appendix II: Standards for Mathematical Practice Grade-Span Descriptions: Pre-K–5, 6–8, 9–12

## Standards for Mathematical Practice Grades Pre-K–5

### 1. Make sense of problems and persevere in solving them.

Mathematically proficient elementary students explain to themselves the meaning of a problem, look for entry points to begin work on the problem, and plan and choose a solution pathway. For example, young students might use concrete objects or pictures to show the actions of a problem, such as counting out and joining two sets to solve an addition problem. If students are not at first making sense of a problem or seeing a way to begin, they ask questions that will help them get started. As they work, they continually ask themselves, “Does this make sense?" When they find that their solution pathway does not make sense, they look for another pathway that does. They may consider simpler forms of the original problem; for example, to solve a problem involving multi-digit numbers, they might first consider similar problems that involve multiples of ten or one hundred. Once they have a solution, they look back at the problem to determine if the solution is reasonable and accurate. They often check their answers to problems using a different method or approach. Mathematically proficient students consider different representations of the problem and different solution pathways, both their own and those of other students, in order to identify and analyze correspondences among approaches. They can explain correspondences among physical models, pictures, diagrams, equations, verbal descriptions, tables, and graphs.

### 2. Reason abstractly and quantitatively.

Mathematically proficient elementary students make sense of quantities and their relationships in problem situations. They can contextualize quantities and operations by using images or stories. They interpret symbols as having meaning, not just as directions to carry out a procedure. Even as they manipulate the symbols, they can pause as needed to access the meaning of the numbers, the units, and the operations that the symbols represent. Mathematically proficient students know and flexibly use different properties of operations, numbers, and geometric objects. They can contextualize an abstract problem by placing it in a context they then use to make sense of the mathematical ideas. For example, when a student sees the expression 40-26, the student might visualize this problem by thinking, if I have 26 marbles and Marie has 40, how many more do I need to have as many as Marie? Then, in that context, the student thinks, 4 more will get me to a total of 30, and then 10 more will get me to 40, so the answer is 14. In this example, the student uses a context to think through a strategy for solving the problem, using the relationship between addition and subtraction and decomposing and recomposing the quantities. The student then uses what he/she did in the context to identify the solution of the original abstract problem. Mathematically proficient students can also make sense of a contextual problem and express the actions or events that are described in the problem using numbers and symbols. If they work with the symbols to solve the problem, they can then interpret their solution in terms of the context.

### 3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient elementary students construct verbal and written mathematical arguments—that is, explain the reasoning underlying a strategy, solution, or conjecture—using concrete referents such as objects, drawings, diagrams, and actions. Arguments may also rely on definitions, previously established results, properties, or structures. For example, a student might argue that two different shapes have equal area because it has already been demonstrated that both shapes are half of the same rectangle. Students might also use counterexamples to argue that a conjecture is not true—for example, a rhombus is an example that shows that not all quadrilaterals with 4 equal sides are squares; or, multiplying by 1 shows that a product of two whole numbers is not always greater than each factor. Mathematically proficient students present their arguments in the form of representations, actions on those representations, and explanations in words (oral or written). In the elementary grades, arguments are often a combination of all three. Some of their arguments apply to individual problems, but others are about conjectures based on regularities they have noticed across multiple problems (see MP.8). As they articulate and justify generalizations, students consider to which mathematical objects (numbers or shapes, for example) their generalizations apply. For example, young students may believe a generalization about the behavior of addition applies to positive whole numbers less than 100 because those are the numbers with which they are currently familiar. As they expand their understanding of the number system, they may reexamine their conjecture for numbers in the hundreds and thousands. In upper elementary grades, students return to their conjectures and arguments about whole numbers to determine whether they apply to fractions and decimals. Mathematically proficient students can listen to or read the arguments of others, decide whether they make sense, ask useful questions to clarify or improve the arguments, and build on those arguments. They can communicate their arguments both orally and in writing, compare them to others, and reconsider their own arguments in response to the critiques of others.

### 4. Model with mathematics.

When given a problem in a contextual situation, mathematically proficient elementary students can identify the mathematical elements of a situation and create or interpret a mathematical model that shows those elements and relationships among them. The mathematical model might be represented in one or more of the following ways: numbers and symbols; geometric figures, pictures, or physical objects used to abstract the mathematical elements of the situation; a mathematical diagram such as a number line, table, or graph; or students might use more than one of these to help them interpret the situation. For example, when students encounter situations such as sharing a pan of cornbread among six people, they might first show how to divide the cornbread into six equal pieces using a picture of a rectangle. The rectangle divided into 6 equal pieces is a model of the essential mathematical elements of the situation. When the students learn to write the name of each piece in relation to the whole pan as 1 experiments, and observational 6, they are now modeling the situation with mathematical notation. Mathematically proficient students are able to identify important quantities in a contextual situation and use mathematical models to show the relationships of those quantities, particularly in multi-step problems or problems involving more than one variable. For example, if there is a penny jar that starts with three pennies in the jar, and four pennies are added each day, students might use a table to model the relationship between number of days and number of pennies in the jar. They can then use the model to determine how many pennies are in the jar after 10 days, which in turn helps them model the situation with the expression, 4 x 10 + 3. Mathematically proficient students use and interpret models to analyze relationships and draw conclusions. They interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. As students model situations with mathematics, they are choosing tools appropriately (MP.5). As they decontextualize the situation and represent it mathematically, they are also reasoning abstractly (MP.2).

### 5. Use appropriate tools strategically.

Mathematically proficient elementary students consider the tools that are available when solving a mathematical problem, whether in a real-world or mathematical context. These tools might include physical objects (cubes, geometric shapes, place value manipulatives, fraction bars, etc.); drawings or diagrams (number lines, tally marks, tape diagrams, arrays, tables, graphs, etc.); models of mathematical concepts, paper and pencil, rulers and other measuring tools, scissors, tracing paper, grid paper, virtual manipulatives, appropriate software applications, or other available technologies. Examples: a student may use graph paper to find all the possible rectangles that have a given perimeter or use linking cubes to represent two quantities and then compare the two representations side by side. Proficient students are sufficiently familiar with tools appropriate for their grade and areas of content to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained from their use as well as their limitations. Students choose tools that are relevant and useful to the problem at hand. These include tools mentioned above, as well as mathematical tools such as estimation or a particular strategy or algorithm. For example, in order to solve 3∕5 – ½, a student might recognize that knowledge of equivalents of ½ is an appropriate tool: since ½ is equivalent to 2½ fifths, the result is ½ of a fifth or 1∕10. This practice is also related to looking for structure (MP.7), which often results in building mathematical tools that can then be used to solve problems.

### 6. Attend to precision.

Mathematically proficient elementary students communicate precisely to others both verbally and in writing. They start by using everyday language to express their mathematical ideas, realizing that they need to select words with clarity and specificity rather than saying, for example, “it works" without explaining what “it" means. As they encounter the ambiguity of everyday terms, they come to appreciate, understand, and use mathematical vocabulary. Once young students become familiar with a mathematical idea or object, they are ready to learn more precise mathematical terms to describe it. In using mathematical representations, students use care in providing appropriate labels to precisely communicate the meaning of their representations. When making mathematical arguments about a solution, strategy, or conjecture (see MP.3), mathematically proficient students learn to craft careful explanations that communicate their reasoning by referring specifically to each important mathematical element, describing the relationships among them, and connecting their words clearly to their representations. Elementary students use mathematical symbols correctly and can describe the meaning of the symbols they use. When measuring, mathematically proficient students use tools and strategies to minimize the introduction of error. Mathematically proficient students specify units of measure; label charts, graphs, and drawings; calculate accurately and efficiently; and use clear and concise notation to record their work.

### 7. Look for and make use of structure.

Mathematically proficient elementary students use structures such as place value, the properties of operations, other generalizations about the behavior of the operations (for example, even numbers can be divided into 2 equal groups and odd numbers, when divided by 2, always have 1 left over), and attributes of shapes to solve problems. In many cases, they have identified and described these structures through repeated reasoning (MP.8). For example, when younger students recognize that adding 1 results in the next counting number, they are identifying the basic structure of whole numbers. When older students calculate 16 x 9, they might apply the structure of place value and the distributive property to find the product: 16 x 9 = (10 + 6) x 9 = (10 x 9) + (6 x 9). To determine the volume of a 3 x 4 x 5 rectangular prism, students might see the structure of the prism as five layers of 3 x 4 arrays of cubes.

### 8. Look for and express regularity in repeated reasoning.

Mathematically proficient elementary students look for regularities as they solve multiple related problems, then identify and describe these regularities. For example, students might notice a pattern in the change to the product when a factor is increased by 1: 5 x 7 = 35 and 5 x 8 = 40—the product changes by 5; 9 x 4 = 36 and 10 x 4 = 40—the product changes by 4. Students might then express this regularity by saying something like, “When you change one factor by 1, the product increases by the other factor." Younger students might notice that when tossing two-color counters to find combinations of a given number, they always get what they call “opposites"—when tossing 6 counters, they get 2 red, 4 yellow and 4 red, 2 yellow and when tossing 4 counters, they get 1 red, 3 yellow and 3 red, 1 yellow. Mathematically proficient students formulate conjectures about what they notice, for example, that when 1 is added to a factor, the product increases by the other factor. As students practice articulating their observations both verbally and in writing, they learn to communicate with greater precision (MP.6). As they explain why these generalizations must be true, they construct, critique, and compare arguments (MP.3).

## Standards for Mathematical Practice Grades 6–8

### 1. Make sense of problems and persevere in solving them.

Mathematically proficient middle school students set out to understand a problem and then look for entry points to its solution. They analyze problem conditions and goals, translating, for example, verbal descriptions into mathematical expressions, equations, or drawings as part of the process. They consider analogous problems, and try special cases and simpler forms of the original in order to gain insight into its solution. For example, to understand why a 20% discount followed by a 20% markup does not return an item to its original price, they might translate the situation into a tape diagram or a general equation; or they might first consider the result for an item priced at $1.00 or $10.00. Mathematically proficient students can explain how alternate representations of problem conditions relate to each other. For example, they can navigate among tables, graphs, and equations representing linear relationships to gain insights into the role played by constant rate of change. Mathematically proficient students check their answers to problems and they continually ask themselves, “Does this make sense?” and “Can I solve the problem in a different way?” While working on a problem, they monitor and evaluate their progress and change course if necessary. They can understand the approaches of others to solving complex problems and compare approaches.

### 2. Reason abstractly and quantitatively.

Mathematically proficient middle school students make sense of quantities and relationships in problem situations. For example, they can apply ratio reasoning to convert measurement units and proportional relationships to solve percent problems. They represent problem situations using symbols and then manipulate those symbols in search of a solution (decontextualize). They can, for example, solve problems involving unit rates by representing the situations in equation form. Mathematically proficient students also pause as needed during problem solving to double-check the meaning of the symbols involved. In the process, they can look back at the applicable units of measure to clarify or inform solution steps (contextualize). Students can integrate quantitative information and concepts expressed in text and visual formats. Quantitative reasoning also entails knowing and flexibly using different properties of operations and objects. For example, in middle school, students use properties of operations to generate equivalent expressions and use the number line to understand multiplication and division of rational numbers.

### 3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient middle school students understand and use assumptions, definitions, and previously established results in constructing verbal and written arguments. They make and explore the validity of conjectures. They can recognize and appreciate the use of counterexamples, for example, using numerical counterexamples to identify common errors in algebraic manipulation, such as thinking that 5 - 2x is equivalent to 3x. Mathematically proficient students can explain and justify their conclusions using numerals, symbols, and visuals, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. For example, they might argue that the great variability of heights in their class is explained by growth spurts, and that the small variability of ages is explained by school admission policies. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and if there is a flaw in an argument, explain what it is. Students engage in collaborative discussions, drawing on evidence from texts and arguments of others, follow conventions for collegial discussions, and qualify their own views in light of evidence presented. They consider questions such as “How did you get that?” “Why is that true?” and “Does that always work?”

### 4. Model with mathematics.

Mathematically proficient middle school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. This might be as simple as translating a verbal or written description to a drawing or mathematical expression. It might also entail applying proportional reasoning to plan a school event or using a set of linear inequalities to analyze a problem in the community. Mathematically proficient students are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. For example, they can roughly fit a line to a scatter plot to make predictions and gather experimental data to approximate a probability. They are able to identify important quantities in a given relationship such as rates of change and represent situations using such tools as diagrams, tables, graphs, flowcharts and formulas. They can analyze their representations mathematically, use the results in the context of the situation, and then reflect on whether the results make sense while possibly improving the model.

### 5. Use appropriate tools strategically.

Mathematically proficient middle school students strategically consider the available tools when solving a mathematical problem and while exploring a mathematical relationship. These tools might include pencil and paper, concrete models, a ruler, a protractor, a graphing calculator, a spreadsheet, a statistical package, or dynamic geometry software. Proficient students make sound decisions about when each of these tools might be helpful, recognizing both the insights to be gained and their limitations. For example, they use estimation to check reasonableness, graphs to model functions, algebra tiles to see how properties of operations apply to algebraic expressions, graphing calculators to solve systems of equations, and dynamic geometry software to discover properties of parallelograms. When making mathematical models, they know that technology can enable them to visualize the results of their assumptions, to explore consequences, and to compare predictions with data. Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

### 6. Attend to precision.

Mathematically proficient middle school students communicate precisely to others both verbally and in writing. They present claims and findings, emphasizing salient points in a focused, coherent manner with relevant evidence, sound and valid reasoning, well-chosen details, and precise language. They use clear definitions in discussion with others and in their own reasoning and determine the meaning of symbols, terms, and phrases as used in specific mathematical contexts. For example, they can use the definition of rational numbers to explain why a number is irrational and describe congruence and similarity in terms of transformations in the plane. They state the meaning of the symbols they choose, consistently and appropriately, such as inputs and outputs represented by function notation. They are careful about specifying units of measure, and label axes to display the correct correspondence between quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate to the context. For example, they accurately apply scientific notation to large numbers and use measures of center to describe data sets.

### 7. Look for and make use of structure.

Mathematically proficient middle school students look closely to discern a pattern or structure. They might use the structure of the number line to demonstrate that the distance between two rational numbers is the absolute value of their difference, ascertain the relationship between slopes and solution sets of systems of linear equations, and see that the equation 3x = 2y represents a proportional relationship with a unit rate of 3/2 = 1.5. They might recognize how the Pythagorean Theorem is used to find distances between points in the coordinate plane and identify right triangles that can be used to find the length of a diagonal in a rectangular prism. They also can step back for an overview and shift perspective, as in finding a representation of consecutive numbers that shows all sums of three consecutive whole numbers are divisible by six. They can see complicated things as single objects, such as seeing two successive reflections across parallel lines as a translation along a line perpendicular to the parallel lines or understanding 1.05*a* as an original value, *a*, plus 5% of that value, 0.05*a*.

### 8. Look for and express regularity in repeated reasoning.

Mathematically proficient middle school students notice if calculations are repeated, and look for both general methods and shortcuts. Working with tables of equivalent ratios, they might deduce the corresponding multiplicative relationships and make generalizations about the relationship to rates. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1; 2) with slope 3, students might abstract the equation (y – 2)∕(x – 1) = 3. Noticing the regularity with which interior angle sums increase with the number of sides in a polygon might lead them to the general formula for the interior angle sum of an *n*-gon. As they work to solve a problem, mathematically proficient students maintain oversight of the process while attending to the details. They continually evaluate the reasonableness of their intermediate results.

## Standards for Mathematical Practice Grades 9–12

### 1. Make sense of problems and persevere in solving them.

Mathematically proficient high school students analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. High school students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph and interpret representations of data, and search for regularity or trends. Mathematically proficient students check their answers to problems using different methods of solving, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

### 2. Reason abstractly and quantitatively.

Mathematically proficient high school students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically, and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students can write explanatory text that conveys their mathematical analyses and thinking, using relevant and sufficient facts, concrete details, quotations, and coherent development of ideas. Students can evaluate multiple sources of information presented in diverse formats (and media) to address a question or solve a problem. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meanings of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

### 3. Construct viable arguments and critique the reasoning of others.

Mathematically proficient high school students understand and use stated assumptions, definitions, and previously established results in constructing verbal and written arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples and specific textual evidence to form their arguments. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is and why. They can construct formal arguments relevant to specific contexts and tasks. High school students learn to determine domains to which an argument applies. Students listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. Students engage in collaborative discussions, respond thoughtfully to diverse perspectives and approaches, and qualify their own views in light of evidence presented.

### 4. Model with mathematics.

Mathematically proficient high school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

### 5. Use appropriate tools strategically.

Mathematically proficient high school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for high school to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. They are able to use technological tools to explore and deepen their understanding of concepts. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems.

### 6. Attend to precision.

Mathematically proficient high school students communicate precisely to others both verbally and in writing, adapting their communication to specific contexts, audiences, and purposes. They develop the habit of using precise language, not only as a mechanism for effective communication, but also as a tool for understanding and solving problems. Describing their ideas precisely helps students understand the ideas in new ways. They use clear definitions in discussions with others and in their own reasoning. They state the meaning of the symbols that they choose. They are careful about specifying units of measure, labeling axes, defining terms and variables, and calculating accurately and efficiently with a degree of precision appropriate for the problem context. They develop logical claims and counterclaims fairly and thoroughly in a way that anticipates the audiences’ knowledge, concerns, and possible biases. High school students draw specific evidence from informational sources to support analysis, reflection, and research. They critically evaluate the claims, evidence and reasoning of others and attend to important distinctions with their own claims or inconsistencies in competing claims. Students evaluate the conjectures and claims, data, analysis, and conclusions in texts that include quantitative elements, comparing those with information found in other sources.

### 7. Look for and make use of structure.

Mathematically proficient high school students look closely to discern a pattern or structure. In the expression *x*2 + 9*x* + 14, high school students can see the 14 as 2 × 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see 5 – 3(*x* – *y*)2 as 5 minus a positive number times a square, and use that to realize that its value cannot be more than 5 for any real numbers *x* and *y*.

### 8. Look for and express regularity in repeated reasoning.

Mathematically proficient high school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms sum to zero when expanding (*x* – 1)(*x* + 1), (*x* – 1)(*x*2+ *x* + 1), and (*x* – 1)(*x*3 + *x*2 + *x* + 1) might lead students to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details and continually evaluating the reasonableness of their intermediate results.