Guidelines for the Mathematical Preparation of Elementary Teachers

July 2007

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MEMORANDUM

TO: Superintendents, Principals, Teachers, Elementary Level Teacher Preparation Programs, Higher Education Mathematics Faculty, Education Associations, and Candidates for Licenses at the Elementary Level

FROM: David P. Driscoll, Commissioner of Education

DATE: 31 July 2007

SUBJECT: Guidelines for the Mathematical Preparation of Teachers at the Elementary Level

I am pleased to provide you with "Guidelines for the Mathematical Preparation of Elementary Teachers." These Guidelines are called for in recent amendments to the regulations for educator preparation and licensure.

As you know, we must prepare teachers and students for a world that is increasingly technological and globally competitive. These new realities demand far higher levels of proficiency in the STEM disciplines: Science, Technology, Engineering and Mathematics. Mathematics is the foundation for accomplishment in all four of these disciplines, and Massachusetts, despite making great strides, still has a long way to go before every student is proficient in math.

These Guidelines highlight the breadth and depth of mathematics that teachers at the elementary level must not only be able to do, but understand and explain in many ways to students. The fact that the Guidelines do not extend beyond the mathematics covered by elementary schools should not be construed to mean that the mathematics preparation recommended herein will be easy—quite the contrary. It will require most candidates to delve far deeper into the underlying structures of mathematics than they have previously explored. It will require mathematics and teacher preparation program faculty to substantially rethink and redesign their courses.

I invite those interested in teaching at the elementary level to take the time to read these Guidelines and understand the breadth and depth of mathematics they need to be successful in this role. I have already invited mathematics and teacher preparation faculty to contribute to these Guidelines and they are much improved because of their contributions. I now urge mathematics faculty to tackle the important work of developing a strong mathematics foundation in our next generation of teachers with the same vitality that you have devoted to mathematics itself. We share a common interest in moving from just a tenth of our society truly proficient in mathematics to one where this foundation is universal. This transition must begin in our elementary classrooms! Finally, I ask that mathematics and teacher preparation faculty form even stronger partnerships in this endeavor so that this deeper mastery of math is successfully translated into more effective teaching of math.

My hope and expectation is that these Guidelines will be a living document. Please share what you learn from their implementation and how they can be improved via letter or e-mail to nperkins@doe.mass.edu.

Acknowledgement

I want to thank the many people and organizations that contributed to these Guidelines, including many prominent mathematicians from across the nation. We are particularly indebted to the principal author, Tom Fortmann, who has spent the last decade laying the groundwork for this advance in the mathematics preparation of elementary teachers. Massachusetts mathematics professors Richard Bisk and Solomon Friedberg, as well as Dr. Andrew Chen, have been particularly instrumental in the success of this work. Now that Tom Fortmann has joined the Massachusetts Board of Education, we can look forward to his continuing contributions to excellence in education.

Guidelines for the Mathematical Preparation of Elementary Teachers

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1. Context and Purpose

Elementary teachers are the front line of mathematics education, preparing students, parents, workers, *and future teachers* across the Commonwealth for the secondary grades, for college, and for careers that require increasingly demanding levels of mathematical skill and thinking. Along with some very engaged and skilled parents, a subset of our elementary teachers laid the mathematical foundations for most of our current mathematicians, scientists, and engineers. However, only an estimated 10% of our adult population is fluent enough in mathematics to consider pursuing such careers, and an alarming number do not have the math skills needed for entry-level jobs.

The purpose of these Guidelines is to strengthen the mathematics preparation of teachers at the elementary level as called for by the Massachusetts Board of Education. Most elementary teachers have not had sufficient mathematics content-knowledge preparation for their critical role (see [1-9] and other national publications). Our students' math achievement, ahead of the nation but far below that of their international peers, will not rise until mathematics teaching and learning improves vastly—starting with elementary school.

Accordingly, the Massachusetts Board of Education added the following Subject Matter Knowledge Requirement for elementary teachers to regulation 603 CMR 7.06(7)(b) on April 24, 2007 (emphasis added):

> 2. Mathematics.
>
> a. Basic principles and concepts important for teaching elementary-school mathematics in the following areas.
>   i. Number and operations (the foundation of areas ii-iv)
>   ii. Functions and algebra
>   iii. Geometry and measurement
>   iv. Statistics and probability
>
> b. Candidates shall demonstrate that they possess both fundamental computation skills and comprehensive, in-depth understanding of K–8 mathematics. They must demonstrate not only that they know *how to do* elementary mathematics, but that they *understand* and can explain to students, in multiple ways, *why it makes sense*.
>
> c. The Commissioner, in consultation with the Chancellor of Higher Education, **shall issue guidelines for the scope and depth of knowledge expected in mathematics**, described in a. and b. above.

The Board has also directed that beginning January 2009, the Massachusetts Tests for Educator Licensure (MTEL) "General Curriculum Test"[1] will include a separately scored section of 40 questions on the mathematics specified in the new regulation. This document comprises the Commissioner's Guidelines, articulating the scope and depth of mathematics knowledge—*both skills and understanding*—that are expected of elementary teachers and that will be assessed on the test.

[1] Candidates for elementary teaching licenses must pass three MTEL examinations: General Curriculum, "Communication & Literacy Skills" and "Foundations of Reading." The current General Curriculum Test covers Language Arts (18 questions), History/Social Science (18), Mathematics (18), Science (18), and Child Development (14). A single passing score is required, i.e., scoring across the five subjects is "compensatory." The new test will require passing scores on two subtests, one covering Language Arts (18 questions), History/Social Science (18), Science (18), and the other covering mathematics (40).

These Guidelines have been developed with three audiences in mind:

A. Mathematics Department Faculty

We—the candidates, the teacher preparation programs, and the Commonwealth—need *you* to be teaching the courses referenced by these Guidelines. It is no longer someone else's problem that so many students and, ultimately, members of the workforce are ill-prepared for the challenges of an increasingly technological and competitive world economy. With each passing year, our economy and our quality of life depend upon higher levels of proficiency in the "STEM" disciplines (Science, Technology, Engineering and Mathematics) across the workforce.

Our Commonwealth is not yet on this path. You don't need to be told about the weak mathematics preparation of many college students—you see it every day in your classrooms. This new regulation addresses a national problem that begins in elementary school and accumulates like a snowball grade by grade. Mathematics faculty are uniquely positioned to break the cycle of failure and low expectations because you teach the next generation of teachers. Your success will overcome the aforementioned "10% barrier," providing our state with the STEM expertise it urgently needs and your own departments with more and better prepared candidates.

Various distinguished university mathematicians, including at least three members of the National Academy of Sciences, are actively involved in K–12 math education nationwide and participated in the 2005 Mathematical Sciences Research Institute workshop summarized in reference [4]. This indicates the importance of the task, the role that university mathematicians can play, and the sophistication required to teach mathematics to preservice teachers.

Here in Massachusetts, a growing number of math faculty are teaching both preservice and in-service teachers. One of them commented as follows:

> *As a mathematics professor, I'm all too familiar with the mathematical deficiencies and fears of many of our students. These licensure changes provide an opportunity to deal positively with the problem where it often begins, in the elementary classroom. I used to think that mathematics for elementary teachers was somehow easier and less serious. Having taught a wide range of courses for math majors, I find this work is just as intellectually stimulating and professionally satisfying.*

The goal is not just *more* mathematics but the *right* mathematics, focused on the elementary classroom. These potential teachers must develop *deep understanding* of the math they will teach, and mathematics faculty, hopefully in partnership with teacher preparation faculty, have the background to inspire and empower them—a task that is intellectually challenging, professionally rewarding, and fun. Raising *their* students' math achievement—and readiness for college courses—depends upon *you* stepping up to this important challenge. That's why these Guidelines have been written with you as our "primary audience."

B. Candidates for Licenses at the Elementary Level[2]

These Guidelines will help you understand what the regulations mean by "Candidates shall demonstrate that they possess both fundamental computation skills and comprehensive, in-depth understanding of K–8 mathematics." In this document you may encounter examples that are very challenging and difficult to understand. If so, you are likely among the many candidates for elementary licenses who have far stronger knowledge and skills in language arts than in mathematics and who we hope will benefit from these new requirements.

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[2] Elementary Teacher, Teacher of Students with Moderate Disabilities, Teacher of Students with Severe Disabilities, Teacher of the Deaf and Hard-of-Hearing, and Teacher of the Visually Impaired.

Many of your colleagues have already completed courses that piloted the content/curricula outlined in these Guidelines and found that they have far more capability in mathematics than they gave themselves credit for in the past. We are confident that not only you, but your future generations of students, will achieve the higher levels of mathematics proficiency we all need for an increasingly interconnected and technological world.

The Guidelines include examples of questions similar to those you may find on the revised mathematics section of the General Curriculum Test. Teacher preparation programs and professional development courses will be redesigned to help teacher candidates and in-service teachers gain the mathematical content knowledge that is outlined in the regulations and specified in these Guidelines.

C. Coordinators and Faculty from Approved Preparation Programs for Teachers at the Elementary Level

The combined efforts of educator preparation and mathematics faculty will ensure that students benefit from the partnerships envisioned by these Guidelines—partnerships already begun among most preparation programs and schools in the arts and sciences, and nurtured by activities such as the workshop summarized in reference [4].

Teacher preparation faculty continue to play a vital role in helping candidates transform their subject-matter knowledge into competent teaching of mathematics to a diverse student body. The difference is that when fully implemented, these Guidelines will end the era of attempting to master the teaching of a subject that has not itself been mastered by the teacher!

Sources

The level and nature of coursework described herein is based upon the advice of mathematicians with considerable experience teaching and assessing preservice and in-service elementary teachers, and it draws upon the following national and state recommendations:

[1] The National Council of Teachers of Mathematics (NCTM) recommends at least three college math courses for elementary teachers, emphasizing mathematical structures essential to those grades.

[2] The Conference Board of the Mathematical Sciences (CBMS) recommends at least three college math courses for K–4 teachers, seven courses for 5–8, and teaching by math specialists starting in grade 5; it also discusses at length the scope and depth of those courses.

[3] A U.S. Department of Education (USDE/FIE) advisory committee recommends at least four college math courses for teachers in grades K–8 and describes those courses in detail.

[4] The Mathematical Sciences Research Institute (MSRI) summarizes recommendations from a workshop on the mathematical knowledge needed for teaching in grades K–8.

[5] The Massachusetts Great Schools Campaign will recommend 3–4 college math courses for elementary teachers in a forthcoming report.

Full citations are provided in the bibliography at the end of this document, where all of the references [1–18] provide context and guidance for anyone designing or revising courses for preservice teachers at any grade level. Also listed are current and forthcoming textbooks for this purpose [T1–T12].

2. Number of Courses

Most approved programs for teaching licenses at the elementary level will need to expand the number and depth of mathematics courses that are available to their candidates. As in every subject area, candidates will have developed different levels of competence in mathematics prior to enrolling in the program. However, the research is clear that competence across the population in general, including candidates for licenses at the elementary level, is lower in mathematics than in reading, writing, and language arts.

For those candidates enrolling with typical knowledge and fluency in mathematics, attaining the necessary level of content knowledge will normally require at least three to four college-level, subject-matter courses, i.e., 9–12 semester-hours, taught by mathematics faculty, potentially in partnership with education faculty. These should be taken after any necessary remedial courses and either integrated with or taken prior to math methods courses. The Department's program approval staff will require strong justification from programs that propose less than 9 semester-hours for most candidates. Exceptions may be made for candidates with substantial prior math background; education and mathematics faculty should collaborate in identifying such individuals.

Colleges and universities around the Commonwealth have different course structures, schedules, general education requirements, and other constraints. Their students arrive with varying degrees of mathematical preparation and some students must take remedial mathematics courses upon entry.[3] Consequently, rather than attempt to define specific courses, the Department recommends these relative weightings for the four strands:[4]

i.	Number & Operations	45%	iii.	Geometry & Measurement	20%
ii.	Functions & Algebra	25%	iv.	Statistics & Probability	10%

The emphasis on Number & Operations reflects its central role in K–8 mathematics, elementary teachers' well-documented difficulties with this strand, and the other three strands' heavy dependence on it. Although expert opinions differ on the percentages listed above, there is considerable consensus that most students require more study of Number & Operations than will fit into a standard 3 semester-hour course.

It is not necessary to devote a single course to each strand. Indeed, integrating Number & Operations with Functions & Algebra can be very desirable and would fit well into a two-semester sequence, with a third semester devoted to a combination of Geometry & Measurement with Statistics & Probability. A fourth, capstone course focusing on applications of mathematics is recommended and an example appears below.

Prerequisites

Because many students arrive from high school poorly prepared for college mathematics, it is crucial that all preservice teachers obtain the appropriate prerequisite knowledge before beginning these new courses. To this end, incoming freshmen should take the Accuplacer® tests in *both* Arithmetic and Elementary Algebra, or some equivalent placement test, with those who do not pass required to take remedial courses prior to entering an approved program. Alternatively, colleges that require the SAT for admission may choose to use its math component—with an appropriately high cut score—to determine whether remedial courses are necessary. However, such norm-referenced tests cannot answer whether students have mastered the content and depth of understanding addressed by these Guidelines; that task requires diagnostic testing aligned with these standards.

Accuplacer is efficient and inexpensive, and the Elementary Algebra test is already required by all community colleges, state colleges, and the University of Massachusetts. The Arithmetic test is needed as well because so many students slip through high-school algebra without mastery of K–8 arithmetic. The Boards of Education and Higher Education will explore making these tests available in high schools so that students intending to enter approved programs can be certain that they are prepared before entering college.

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[3] The courses described herein are exclusive of any such remedial courses.

[4] This terminology is consistent with that of college and university math departments; the same strands appear in the *Massachusetts Mathematics Curriculum Framework* [15] and the NCTM standards documents [13, 14] with slightly different names.

3. Mathematical Depth

Among the population at large, including many elementary teachers, mathematics is widely perceived as a vast hodgepodge of memorized facts and procedures that don't make much sense. The overarching goal of these courses is to lay that misconception to rest, replacing it with the realization that elementary-school mathematics is a coherent and unified set of concepts and principles that is at once powerful, beautiful, and fun. True conceptual understanding renders rote memorization—the bane of school mathematics—largely unnecessary.

Elementary teacher candidates are expected to attain proficiency with, as well as deep understanding of, the arithmetic, algebra, geometry, and probability that their own students will be expected to master in grades K–8. They can reach this level of knowledge if and only if they

1. Come to view arithmetic (and algebra) as a small, unified, coherent, consistent subject *that all makes sense.* [9; 3, Sec. 3.3]
2. Appreciate the importance of developing clear, explicit, grade-appropriate *definitions* and using *logical reasoning* to arrive at unambiguous conclusions. [9; 7; 4; 3, Secs. 3.1–3.5]
3. Experience and do *real mathematics*, by struggling with problems that have multiple steps, logical challenges, and non-obvious solutions. [11, 4]
4. Acquire *habits of mathematical thinking:* reasoning, conjecturing, visualizing, analyzing, estimating, exploring, justifying, and constantly probing with "Why?" [10; 2, pp. 8, 13–14]
5. Traverse many *levels of abstraction:* from marks on a wall to Roman numerals to place value to scientific notation; from numbers to variables (a central abstraction of algebra) to functions. [3, Sec. 3.4]
6. Gain the competence *and confidence* to analyze their students' mathematical thinking and engage them in productive mathematical discourse. [7]

CBMS [2] explores these themes, applauds the "conceptual richness of early content," and provides an interesting perspective on the role of mathematicians:

> *In taking responsibility for the mathematics education of elementary teachers, mathematicians are invited, in effect, to re-enter the world of the naïve mathematical thinker. The recognition that the "unsophisticated" questions teachers pose do raise fundamental issues should inspire instructors to find contexts in which these can be addressed fruitfully. This means, at least initially, approaching the mathematics from an experientially based direction, rather than an abstract/deductive one. Isn't this the way each of us starts our individual journey into the world of mathematics?* [2, p. 95]

Mathematics professors should realize that these are in no sense "remedial" courses and that imparting the required depth of mathematical understanding to teachers constitutes just as great an intellectual challenge as teaching more abstract subjects to math majors. Teachers need to understand, for example, how the distributive and other properties govern all of arithmetic and lead on to algebra; that subtraction and division are the "inverses" of addition and multiplication; that place value is the cornerstone of modern mathematics, science, and technology; and that proportions are instances of linear functions. Every capable instructor of K–8 teachers soon recognizes that *elementary mathematics is not elementary*.

Definitions and Reasoning

Various mathematicians stress the paramount importance of precise definitions and sound mathematical reasoning, lamenting the conspicuous absence of those ingredients at key points in most K–12 curricula. The U.S. Department of Education devotes an entire chapter [3, Chap. 3] to this issue and Wu [9] summarizes it as follows:

> *Mathematics is by its very nature a subject of transcendental clarity. In context, there is never any doubt as to what a concept means, why something is true, or where a certain concept or theorem is situated in the overall mathematical structure. Yet mathematics is often presented to school students as a mystifying mess. No doubt the textbooks are at fault, but many of the teachers certainly contributed their share to the obfuscation. For this reason, we would want teachers to have a firm grasp of the following characteristics of mathematics, namely,*
>
> *that precise definitions form the basis of any mathematical explanation, and without explanations mathematics becomes difficult to learn,*
>
> *that logical reasoning is the lifeblood of mathematics, and one must always ask* why *as well as find out the answer, and finally,*
>
> *that concepts and facts in mathematics are tightly organized as part of a coherent whole so that the understanding of any fact or concept requires also the understanding of its interconnections with other facts and concepts.* [9, p. 2]

This is not to say that definitions must always come first. On the contrary, they should emerge from experience and discussion aimed at building an intuitive model of the thing to be defined,[5] but eventually students must have a clear, unambiguous basis on which to build further knowledge.

Definitions and reasoning must of course be tailored to the students' level of thinking [3, p. 91]. This is where mathematics and pedagogy intersect, as observed by Ball et al.:

> *. . . there are predictable and recurrent tasks that teachers face that are deeply entwined with mathematics and mathematical reasoning—figuring out where a student has gone wrong (error analysis), explaining the basis for an algorithm in words that children can understand and showing why it works (principled knowledge of algorithms and mathematical reasoning), and using mathematical representations. Important to note is that each of these common tasks of teaching involves mathematical reasoning as much as it does pedagogical thinking.*
>
> . . .
>
> *Teachers need skill with mathematical terms and discourse that enable careful mathematical work by students, and that do not spawn misconceptions or errors. Students need definitions that are usable, relying on terms and ideas they already understand. This requires teachers to know more than the definitions they might encounter in university courses.* [7, p. 21]

In other words, teachers need sufficient knowledge to utilize definitions that are both age-appropriate and consistent with the formal mathematical definitions ("upwardly compatible"). Otherwise, definitions will seem to change from grade to grade and give students the impression that mathematics is arbitrary.

Precise definitions are particularly important to help students make sense of the following:

- Fractions have many manifestations and properties that should be unified and clarified by saying explicitly what fractions actually are: *numbers*. (See Appendix B.6.)
- Division is usefully interpreted in various ways (repeated subtraction, partitive, measurement), but unless the teacher also defines it *in terms of multiplication* ($A \div B$ is the solution of $? \times B = A$), it's difficult to make sense of division of fractions or explain why division by zero is impossible. An analogous statement applies to subtraction. (See Appendix B.4.)

[5] Just as in the messy process of actually "doing mathematics," definitions and theorems are often crafted later to clean up the results.

- Remainders in whole-number division must be defined carefully; some useful questions are:

  What do you tell a student who says that $\frac{46}{6} \neq \frac{23}{3}$ *because* $6\overline{)46}^{\,7\text{ R}4}$ *but* $3\overline{)23}^{\,7\text{ R}2}$ *?*

  What is the remainder when 7 is divided by 8?

- "Setting up a proportion" can seem akin to magic unless one understands that proportions are simply instances of *constant rates* and *linear functions.*
- Mathematical definitions of words like *even* and *prime* differ from their everyday use [4]. For example, consider prime number, prime rib, and prime time, or even number, even break, and even score.
- Average is defined in the early grades as an arithmetic mean, which is appropriate for a finite set of numbers such as the heights of all students in a classroom. However, teachers must understand and explain to students that they will soon encounter other definitions; otherwise confusion will arise in problems such as

  I drove to work at 60 mph and home at 40 mph. What was my average speed for the round trip?

  Once a more complete understanding of average is mastered, even the following more diabolical version of the problem elicits a smile rather than an "apparent" but wrong answer:

  A car goes up a hill at 30 mph. How fast must it go back down to average 60 mph for the round trip?

Problem Solving

Mathematicians are often perplexed when "problem solving" appears as a distinct topic of study in K–12 curricula and frameworks; after all, they say, mathematics *is* problem solving! Although problem solving is indeed the *sine qua non* of mathematics, attempts to codify it in memorizable procedures are counterproductive.

Moreover, because of K–12's high incidence of ambiguously defined terms, hidden assumptions, and ephemeral patterns, teachers must be able to formulate *well-posed* problems and recognize *ill-posed* problems with hidden assumptions or insufficient information. An example of the latter is

> *One third of the boys in Mrs. Scott's class want to have a peanut butter sandwich for lunch. One fourth of the girls in Mrs. Scott's class want to have a peanut butter sandwich for lunch. What fraction of the children in Mrs. Scott's class wants to have a peanut butter sandwich for lunch?* [T3, p. 115]

Pattern problems are often ill-posed, with an implicit assumption that there is only one way to continue a given pattern. Teachers should recognize the difference between one possible rule and a uniquely determined rule. For example, the sequence 1, 2, 4, 8 need not continue with 16, 32, …

Sections 3.6–3.14 of [3] explore these issues in depth; the latter section on "Working with Problems for Elementary Teachers" is particularly germane to the courses described herein.
Doing real mathematics involves struggling with substantive problems that don't have obvious solutions; teachers need this experience, at a level they can handle. Assignments should include rich activities and problems that stretch the mind, challenge the intellect, and develop mathematical thinking (in addition to exercises that provide practice and solidify skills). For example:

- Sum of consecutive integers, sum of consecutive odd integers, and their geometric interpretations; word problems with those sums hidden in them.
- Problems that require exploration, conjecture, and perseverance:

  A school has 961 lockers and 961 students. The first student opens every locker, the second student changes every second locker, the third student changes every third locker, and so on. Which lockers are open after every student has passed by?

  Analyze and propose a strategy for the game of NIM. Two players begin with 13 markers and take turns removing either one or two; the player who removes the last marker loses.

- Challenging multi-step word problems:

  Two ladies started walking at sunrise, each from her village to the other's village. They met at noon. The first lady arrived in the second's village at 4:00PM, while the second lady arrived at the first lady's village at 9:00PM. They walked at constant rates. What time was sunrise?

- Counterintuitive word problems:

  Fresh cucumbers contain 99% water by weight. Three hundred pounds of cucumbers are placed in storage but, by the time they are brought to market, it is found that they contain only 98% water by weight. How much do these cucumbers now weigh?

- Wide-open questions that stimulate thinking:

  Estimate the number of barbers in Chicago.

- "Design" negative & fractional exponents to maintain the properties of whole-number exponents.
- Suppose your number system consists only of counters or tally marks. Define "odd" and "even." Prove that odd + odd = even and odd + even = odd. Also prove it algebraically.

4. Course Content

The new regulation, quoted on page 1, first specifies four strands of mathematical content. It requires (see Section 2 above) 9–12 semester-hours of coursework in approximately the following proportions to cover the topics necessary for the teaching of elementary mathematics:

i.	Number & Operations	45%
ii.	Functions & Algebra	25%
iii.	Geometry & Measurement	20%
iv.	Statistics & Probability	10%

The mathematical topics and subtopics in these strands are fairly standard. A list of sources offering abundant advice on appropriate topics, syllabi, course structures, etc., may be found in Appendix A.

The new regulation then sets out a more challenging task that bears repeating here:

> b. Candidates shall demonstrate that they possess both fundamental computation skills and comprehensive, in-depth understanding of K–8 mathematics. They must demonstrate not only that they know *how to do* elementary mathematics, but that they *understand* and can explain to students, in multiple ways, *why it makes sense*.

Some mathematics professors have begun designing and delivering courses that meet the challenge, but considerable additional effort and ingenuity are needed to build on this work. The following sections and Appendix B contain suggestions and recommendations to that end, with emphasis on the "understandings" expected of teachers and the key "sense-making" connections, based on recent experience in math content courses for both preservice and in-service teachers. There is, of course, considerable overlap among strands, and integrated approaches are encouraged.

It is important not to underestimate the magnitude of this task. Most existing courses designed for preservice teachers (especially where only one course is required) are too broad to attain the required depth. Other courses designed for math majors rarely focus on the mathematics of the K–8 classroom or on achieving deep understanding thereof. Thus, in many cases it will be necessary to design new courses and substantially redesign others in order to meet these requirements.

The paramount goal of these courses is for teachers to develop deep understanding of mathematics content. Pedagogy is typically the subject of "methods" courses, but separation of these two symbiotic topics is inefficient. Future teachers will be best served by math professors who integrate mathematical principles, where appropriate, with discussion of how these ideas can play out in classrooms, and by education professors who ensure that methods are thoroughly grounded in content. If the new emphasis on content is accompanied by very goal-oriented collaboration between the two departments—perhaps even team-teaching—the benefits for students throughout K–16 are potentially vast.

I. Number & Operations (45%)

Number and operations is the basis for all other school mathematics; connections and examples from algebra and geometry arise frequently and should be emphasized. Because full comprehension of Number & Operations typically requires more than one semester, and because arithmetic, geometry, and algebra share a rich web of relationships, an integrated course sequence incorporating multiple strands should be considered.

A few concepts and issues merit special attention:

- The subtly powerful invention known as *place value* enables all of modern mathematics, science, and engineering. A thorough understanding removes the mystery from computational algorithms, decimals, estimation, scientific notation, and—later—polynomials.
- Integers, fractions, decimals, percents, and mixed numbers are all *just plain numbers*; they share the same properties and each one has a home on the number line.
- The *number line* is critically important in its own right, both as a connection from numbers and counting to linear measurement and as a medium for interpreting and understanding the four operations.
- *Number sense*, mental math, and estimation are indispensable skills.
- *Nine simple properties* govern all of arithmetic; the distributive property is the glue that binds addition and multiplication; the properties and nested *number systems* (counting, whole, integer, rational, real) wrap arithmetic in a small, coherent package that *all makes sense*. (See Appendix B.9.)
- Subtraction and division are the "inverses" of addition and multiplication, respectively, and their agents are *opposites* and *reciprocals*. *Signed numbers* are to addition and subtraction as *fractions* are to multiplication and division.
- Nearly all numbers have associated *units,* an important point that is often overlooked in the early grades. Units pave the way to learning fractions and they provide crucial guidance for solving problems—especially those involving rates.
- Every *proportion* involves a *constant rate* and is an instance of a *linear function*.

Appendix B addresses these points and others in some detail. It does not specify a syllabus or topic list, but rather highlights key areas where understanding is crucial, identifies issues and misconceptions that frequently cause trouble for both teachers and students, and suggests how to approach and explain them. The expected depth of understanding is illustrated with a variety of sample problems.

II. Functions & Algebra (25%)

Algebra, once considered too advanced for K–6, is now recognized as a gatekeeper subject[6] and emerges in the primary grades. Second graders, for example, should learn that the subtraction problem $5-3=?$ is equivalent to $3+?=5$. Because a key objective for elementary teachers in mathematics is to prepare their students for algebra, they must become proficient and comfortable with algebraic thinking, especially the use of variables and solution of simple equations. They should also build upon the algebra implicit in the base-10 number system.

The following concepts and issues merit special attention:

- Basic algebra *generalizes arithmetic*. Rational expressions illustrate the importance of understanding rational numbers.
- The concept of *variable* is the cornerstone of algebra, critical for translating real-world problems into mathematics, and a huge obstacle for many people.
- The concept of *function* is not just an obstacle but frequently a traumatic experience. One must ease into it gently via linear functions and relate it to the "function machines" in elementary curricula, but ultimately teachers should not be unnerved by problems like this:

  Express the area of a circle as a function of its circumference.

- Understanding of functional relationships can be nurtured with "qualitative graphs" [2, p. 31]; for example:

  Joe began walking to school, stopped to chat, continued on, ran back home to get his lunch, and then ran to school. Sketch a graph of his distance from home versus time.

  Given the shape of a glass and water flowing into it at a constant rate, sketch a plot of the height of water in the glass vs. time.

  Given a graph of a vehicle's speed vs. time, sketch a plot of its distance traveled. Given distance traveled, plot speed.

- A *linear equation* (with no constant term) is an infinite collection of *proportions* whose constant rate is the equation's *slope*.
- Cross-multiplication is a memory aid, not a mathematical principle, that should be used only by those who can explain why it works and correctly solve problems like this:

  $$\frac{x}{2}=\frac{3x}{4}-1$$

- The intimate relations among algebra, geometry, and arithmetic (e.g., triangular numbers, Pythagorean Theorem, linear functions) are important, and the ability to move fluently/comfortably among *equations, tables, and graphs* is paramount. This requires a proper definition of *graph* as the set of all points whose coordinates satisfy an equation (or an inequality) in two variables.

  Sketch graphs of the following: $3x+2y=14$ $y\le 2x-3$

- *Patterns* are ubiquitous (arithmetic/numeric, geometric, linguistic, musical/rhythmic) but teachers need help understanding why they're important. For example, the addition and multiplication tables contain several interesting patterns that ease the burden of memorization. Teachers also need to avoid the ill-posed pattern problems that occur frequently in textbooks (see "Problem Solving" in Section 3 above).

[6] Robert Moses' *Radical Equations: Math Literacy and Civil Rights* (Beacon Press, 2001) intriguingly depicts it as a civil rights issue.

- Expression of word problems algebraically, graphically, and pictorially; transcription of verbal information into symbolic language; and problem-solving in general are all subtle and difficult. See USDE/FIE [3], Sec. 3.6 for an overview; Sec. 3.7 contrasting well- and ill-posed questions; and Sec. 2.5 discussing symbolic manipulation.

A caveat on depth vs. breadth: Experience shows that most preservice and in-service elementary teachers find algebra very challenging conceptually. Thus, it is important to focus on attaining depth and understanding of the basics, especially linear functions, graphs, and their relationship to proportions. Teachers must gain an awareness of quadratic and other nonlinear equations, in order to avoid an exclusively linear view, but these topics are not critical in K–6 and have a lower priority than mastering the fundamental concepts.

III. Geometry & Measurement (20%)

Geometry suffers both from an overabundance of definitions (so that some curricula appear to be one big vocabulary lesson) and from imprecision in some of the most important definitions (notably congruence and similarity). Teachers must realize that vocabulary is needed to describe intuitive spatial concepts, and that precision is essential in that vocabulary. But the purpose of the vocabulary is to enable *mathematical reasoning* about spatial objects, and they must master the precision in order to lead their students through grade-level-appropriate activities that emphasize reasoning and develop spatial intuition without getting bogged down in terminology. This task is not easy and relies upon the teacher's own well-developed spatial intuition.

Measurement is fundamental, linking Geometry with both Number & Operations and Algebra:

- A *number line* appears on every ruler, thermometer, scale, speedometer—and on each axis of a multidimensional plot.
- The *area model* of multiplication provides entrée into length, perimeter, area, volume, and surface area—including how they scale together.

  Starting with only the area formula for a rectangle, derive the area formulas for a right triangle, a non-right triangle, a parallelogram, and a trapezoid.

  Derive the volume formula for a rectangular prism. Use it to find the volume of a triangular prism.

  If you triple the edges of a cube, what happens to its surface area? Its volume?

- *Units* and unit conversions are at the heart of measurement (see Section B.11); multiplication and division operations are an excellent venue for discussing compound units.
- The multifaceted relationships among geometry, measurement, arithmetic, and algebra should be explored and exploited. A key example is the Pythagorean Theorem and its application to measuring distance in the plane.

  Cut out four identical right triangles with sides labeled a, b, and hypotenuse c. Arrange them to form new geometric figures and use your knowledge of area and algebra to prove that $a^2 + b^2 = c^2$.

  Starting with a piece of graph paper and a set of coordinate axes, sketch all the points that satisfy the equation $x^2 + y^2 = 1$. *What geometric figure is this? Explain why.*

- The relationship between perimeter and area is a frequent source of confusion:

  A rectangle has perimeter P and area A.   For a fixed value of P, what values of A are possible?
  For a fixed value of A, what values of P are possible?

- Circles, spheres, and π drive one's thinking beyond linear measurement.

  Measure several circles and determine the ratio of circumference to diameter for each.

  Explain why the volume formulas for cylinders, cones, and spheres make sense.

Mastery of a few key topics in Geometry will serve elementary teachers very well:

- Visualization of objects and motion in two and three dimensions.
- Constructions using straightedge and compass.
- Properties of parallel lines, especially angle relationships on a transversal (the "parallel postulate"). Well-developed visual intuition is more important than memorizing textual rules about alternate interior angles and the like.

- Properties of triangles and other polygons.[7] The definition of a polygon is a subtle matter, including its dual role as a curve and a region. Exercises on angle relationships should illustrate the immense utility of drawing auxiliary lines, e.g.:

  Prove that the (measures of the) interior angles of a triangle sum to 180°.

  Find the sum of the interior angles of an n*-sided polygon.*

- Transformations: rotation, translation, reflection, and dilation.

  What are the effects of transformations on area and volume?

- Precise definitions of congruence and similarity, using transformations, that apply to all shapes (not just polygons).
- Similar triangles and their relationship to linear functions, proportion, and rate.

Topics in Geometry and Measurement provide particularly fertile ground for cultivating deductive logic, definition, and proof, as well as appreciation of their central role in mathematics.

[7] The fact that triangles are rigid but other polygons are not is often overlooked in elementary curricula.

IV. Statistics & Probability (10%)

Statistics and probability in K–6 curricula are simple and mostly descriptive, and the NCTM's recent *Curriculum Focal Points* [16] acknowledge their secondary role prior to middle and high school. Nevertheless, teachers should understand the basic concepts, including a modicum of inferential statistics, and be prepared to discuss them with students.

- Descriptive statistics, graphs, and *how they are used* to summarize data that are too complex to digest in detail. Lines of best fit should be discussed, even if not derived rigorously.
- Measures of central tendency (mean, median, and mode) and dispersion (range, standard deviation, etc.). Teachers must understand each measure, how they differ, and *when their use is appropriate*.[8]
- Permutations, combinations, and their applications in computing probability.

  How many different ways can the letters of "educator" be rearranged?
  Will the same technique work on "Mississippi"? Explain.

  What's the probability of drawing 4 aces in a poker hand of 5 cards?

- Sample space, simple/compound events, independent/dependent events, conditional probability.

  A fair coin comes up heads four flips in a row. What is the probability of tails on the next flip?

  Your opponent draws 4 aces in two hands in a row. What might you infer?

- (Optional) Probability distributions, including the normal distribution, and expected value are beyond the scope of K–6 mathematics, but a brief survey of these concepts is desirable if time permits.

[8] For example, only mode works for qualitative data but mode is ill-behaved (discontinuous) with measured data.

V. Capstone Course

In addition, a fourth course should be included if at all possible, and colleges are encouraged to design capstone courses that link mathematics with science and engineering. Here is a hypothetical example:

Applied Mathematics for Elementary Teachers

This course will link elementary mathematics with science, engineering, technology, finance, and economics. By focusing on how mathematics is applied in science, and how dependent science is on mathematics, it will be equally relevant to teachers of both disciplines. The goals are:

- Consolidate, solidify, and motivate mathematics content knowledge from other courses
- Enable and motivate teachers to undertake substantive science and engineering topics

The course syllabus will consist of a series of in-depth applications, case studies, and projects—each with substantial mathematical content—drawn from physics, biology, chemistry, engineering, earth science, computer technology, finance, etc.

Example applications:

- *Physics*: equations of motion, free fall, orbital dynamics; optics
- *Engineering*: static forces in Big Dig; Acela power train; cell phone system design
- *Chemistry*: spectroscopy & molecular absorption
- *Biology*: population dynamics; genetic information encoding
- *Digital Technology*: binary numbers and logic; Google search algorithm; computer design
- *Earth Science*: global warming; plate tectonics; volcano dynamics
- *Finance*: compound interest, investment analysis, internal rate of return, discount rate
- *Economics*: why printing more money doesn't expand an economy

Ideally, such courses should be designed and taught collaboratively by professors of mathematics, science, engineering, economics, etc.

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[3] USDE/FIE[9] Advisory Committee on Improving Teacher Content Knowledge in Mathematics (R. James Milgram, principal author), *The Mathematics Pre-Service Teachers Need to Know*, 2005 (580 pages). This report recommends at least four college math courses for teachers in grades K-8 and describes those courses in detail. ftp://math.stanford.edu/pub/papers/milgram/FIE-book-high-res.pdf

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[T11] Susan Addington, *Measuring the World*, prepublication draft, 2006.

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Appendix A Topics and Syllabi

Expectations for 9–12 semester-hours of mathematics coursework for preservice elementary teachers are set out above, with more detail in Appendix B below. The following sources provide a variety of recommendations regarding topics, syllabi, and course structures:

- CBMS [2] describes in detail the recommended content and emphasis for each strand. Course designers and instructors should read and utilize the following sections:

i.	Number & Operations	pp. 15–20, 25–30, 58–73, 102–108
ii.	Functions & Algebra	pp. 20–21, 30–32, 73–79, 108–110
iii.	Geometry & Measurement	pp. 21–22, 32–34, 80–87, 110–113
iv.	Statistics & Probability	pp. 23–24, 34–36, 87–93, 113–117

- USDE/FIE [3] describes the "basic topics in K–8 mathematics" in Sections 2.1–2.7:

  1. Place Value and Basic Number Skills
  2. Fractions and Decimals
  3. Ratios, Rates, and Percents
  4. The Core Processes of Mathematics: Symbols, Symbolic Manipulation, Solving Equations
  5. Functions and Equations
  6. Real Measurement and Measurement in Geometry

  Sections 2.8–2.9 outline the two Number & Operations courses (*Basics* and *Fractions*) for teachers; all four courses are described in great detail in Chapters 4, 5, 7, and 8.

- Mathematics departments have begun designing undergraduate courses specifically for preservice K–8 teachers, as well as professional development programs for in-service teachers, and a variety of textbooks have appeared for this purpose [T1–T12]. Of these, Parker & Baldridge [T1], Beckmann [T3], and Gross & Gross [T7] have been in use for some time and are strongly recommended.
- Related course descriptions may be found in many college and university catalogs.
- The booklet "Using Math to Teach Math" [4] summarizes the collective knowledge of a select group of mathematicians and math educators at MSRI's workshop on mathematical knowledge for teaching.
- Howe's essay "Taking Place Value Seriously" [12] is a rich and thoughtful source of material on the ubiquitous role of place value in school mathematics.
- NRC's *Adding It Up* [14] explores proficiency in number and operations at great length. Chapter 10 discusses professional development for teachers.
- Transcending the conventional topic-list format, Charles [13] describes the "understandings" that teachers need, grouped into common themes ("Big Ideas") that provide coherence and tie together multiple topics.
- The *MTEL Test Objectives* [18] specify mathematics topics to be assessed for licensure at various levels.
- The *Massachusetts Mathematics Curriculum Framework* [17] provides useful background and context, but the courses described herein are expected to exceed the K–8 standards in both breadth and depth.

Appendix B Number & Operations

This appendix addresses Number & Operations in some detail. However, it is not intended as a syllabus, course outline, or topic list: college and university mathematicians will want to develop their own courses, and there are numerous syllabi, topic lists, and textbooks available in the references cited above in Appendix A.

Rather, this appendix offers advice distilled from a variety of experiences teaching mathematics to preservice and in-service teachers and to K–12 students. It is intended to

- Highlight crucial concepts and topics where understanding is often absent.
- Identify issues and misconceptions that frequently cause confusion.
- Outline a strategy to rebut the perception that mathematics is a vast hodgepodge of memorized facts and procedures that don't make much sense.
- Describe—and illustrate with sample problems—the level of proficiency and depth of understanding that are expected of all those who teach mathematics in the Commonwealth.

Candidates for licenses at the elementary level may find it useful to review this appendix since it can help them to confirm the extent of their own mastery of the material and to understand and appreciate the courses in mathematics their preparation program may ask them to complete. When it is apparent to the candidates that such courses will be needed, they can look forward to the “demystification” of math and an end to the frustration they may have experienced previously as they attempted to solve problems using rote memorization rather than the deep understanding they can and will develop.

B.1. Foundations

Certain “obvious” definitions and principles may not be obvious to many undergraduates, including:

- “=” is a statement of numerical equality, *not* an action or an operation on a calculator. Inequalities behave similarly.
- A *solution* to an equation is a value of a variable (or variables) that makes the equation true.
- If two numerical expressions are equal, one can be *substituted* for the other in any mathematical formula. People who are variable-phobic have great difficulty using this important principle.
- What constitutes a *logical proof* and how it differs from an illustrative example (e.g., $3 + 5 = 8$ does not prove that the sum of any two odd numbers is even).
- The difference between socially-agreed convention (e.g., multiply before adding) and logical necessity (e.g., negative $\times$ negative = positive).
- The symbol “–” has three distinct uses, designating subtraction, a negative number, or the opposite of a number.

Establishing this groundwork early will pay dividends later.

B.2. Numbers and place value

Providing teachers with some historical perspective is useful. For example, even without a number system, primitive people could count, record tally marks or counters, add, subtract, and distinguish odd from even. Arabs, Hindus, and others developed sophisticated number systems (including zero) long before Europeans, who struggled with Roman numerals and the abacus until zero was introduced in the 13th century.[10] This eventually led to adoption of our modern *place value* (base 10) system, a subtly powerful invention that underlies all of modern mathematics, science, and engineering. Historical context helps teachers understand that place value is neither obvious nor easy—humans took thousands of years to develop it.[11]

The importance of place value in elementary mathematics cannot be overemphasized, but most people comprehend it only superficially. They can identify the ones, tens, and hundreds places but don't really understand place value's utility in enabling efficient forms of representation (decimal fractions, scientific notation) and computation (standard algorithms, mental math, estimation, rounding). Course designers and instructors should seize every opportunity to underscore and elucidate the role of place value, for example:

- Problems like this one (and others involving days/hours/minutes/seconds):

  1 mile + 3 yards + 5 feet + 7 inches = ____ inches

  1 liter + 3 deciliters + 5 centiliters + 7 milliliters = ____ milliliters

  1 thousand + 3 hundreds + 5 tens + 7 ones = ____ tens

- Explaining the operation of both standard and non-standard (e.g., lattice multiplication) algorithms.
- Illustrating that zero is a quantity just like 1, 2, 3,… and that it differs from "nothing"

  Is zero an even number?

- Exercises involving very large and very small numbers, leading to scientific notation.
- Exercises involving other number bases (base 5 is popular) can be quite effective at helping teachers understand both the workings of the base 10 system and their students' struggles to learn it.
- Base-10 numbers are a particular instance of the *polynomials* that students will encounter in algebra.

The essay [12] explores in depth the role of place value in unifying the concepts of school mathematics, from arithmetic to algebra, and will be a useful source for preservice course designers. References [2] and [3] also have quite a bit to say about this topic.

B.3. The number line

The *number line* is very important because it connects numbers with linear measurement. Teachers must help their students become familiar and comfortable with it from an early stage, using it for ordering, comparing, demonstrating operations, etc. When fractions and signed numbers appear on the scene, they should immediately find homes on the number line, establishing their identity as "just more numbers." Not thinking of fractions as numbers is a major conceptual block for many people.

Doing operations using the number line develops valuable intuition, especially if one begins by laying more palpable objects like 10-rods and 1-cubes on it. It is easy to illustrate various interpretations of subtraction, as well as the relationship between addition and subtraction. Multiplication by positive numbers can be interpreted as stretching (> 1) or shrinking (< 1) the line. Dividing on the number line facilitates visualizing what happens when the divisor is greater or less (in magnitude) than the dividend.

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[10] Remarkably, another 400 years passed before rational numbers were considered fully legitimate.

[11] "The greatest calamity in the history of science was the failure of Archimedes to invent positional notation." —C. F. Gauss

The number line also helps to make sense of operations with signed numbers, especially subtracting a negative number. And if multiplication by –1 is associated with a reflection of the line about the origin, the fact that the product of two negatives is positive makes perfect sense.

B.4. Operations

CBMS [2, p. 58] correctly urges teachers to have "a large repertoire of interpretations of addition, subtraction, multiplication and division, and of ways they can be applied." For example, whole-number addition is usually defined with an appeal to intuition (combining groups and counting), while whole-number multiplication is defined as repeated addition or perhaps with an array model. However, repeated addition of fractions requires some finesse, so it's also useful to define multiplication using either the *set model* or the *area model*.[12] The area model also enables easy visualization of the distributive property.

Subtraction is conceptually more complex and has many useful interpretations (take-away, part-whole, difference), as does division (repeated subtraction, partitive, measurement). These should coalesce into grade-appropriate but precise definitions before students begin the transition to integers and rational numbers. Specifically, teachers must be prepared to explain that

- Subtraction is *defined* in terms of addition ($A - B$ is the solution of $? + B = A$)
- Division is *defined* in terms of multiplication ($A \div B$ is the solution of $? \times B = A$)

Teachers must also reiterate frequently that these pairs of "inverse" operations[13] are inextricable, and relate them to the various interpretations mentioned above. Any teacher who truly understands these definitions should have no trouble—as most do—explaining why $6 \div 0$ makes no sense, and that $0 \div 0$ makes no sense *for a different reason*.

Moving beyond whole-number arithmetic (see Sections B.6–B.8. below), course designers differ on whether negative numbers or fractions should be introduced first. Either way, teachers and students must realize at some point that both of these expansions of the number system have the same purpose: to cope with expressions like $3-4$ and $5 \div 2$ that cannot be represented by whole numbers. Moreover, they must realize that the operations are also being extended to handle *all* numbers (counting, whole, integer, rational) consistently, i.e., such that the properties of arithmetic (Section 9) continue to hold.

Once the operations on integers and rational numbers are introduced, teachers with a clear understanding that

- Subtraction is equivalent to adding the *opposite* ($A - B = A + {}^{-}B$)
- Division is equivalent to multiplying the *reciprocal* ($A \div B = A \times \frac{1}{B}$)

will be able to explain and demystify two notoriously difficult topics: subtraction of negative numbers and division of fractions.

The fifth operation, exponentiation, induces great trepidation because of its unfamiliar rules and nonintuitive behavior ("How can 3^0 be equal to 1 ?!"). This creates an opportunity to show teachers how a careful definition (as repeated multiplication, for positive-integer arguments) and logical reasoning (the rules all follow easily from the definition and needn't be memorized) lead to coherent and useful results. Further reasoning reveals how nonpositive and fractional exponents *must* be defined if those same rules are to remain consistent for these new exponents.

[12] See [3, Sec. 3.13; T1, T3, T4, T7].

[13] In the VMI course [T7], these are colorfully called "unaddition" and "unmultiplication."

B.5. Computation algorithms

The standard (and some nonstandard) algorithms for addition, subtraction, multiplication, and division are efficient, reliable, and eminently useful, but they can be opaque, time-consuming, and error-prone when learned only by rote. Thus, it's incumbent on teachers to fully comprehend these procedures and the place-value structure that makes them work. Communicating that comprehension will enable their own students to more easily understand, remember, and correctly use the algorithms.

Computation algorithms are discussed in [12] at length, in CBMS [2, Chaps. 7–8], and in USDE/FIE [3, Secs. 4.10–4.14]. Section 1.8 of the latter also points out that algorithms are an important mathematical topic in their own right, and understanding the standard elementary algorithms has value well beyond their immediate computational utility:

> *It is very important that pre-service teachers understand that algorithms play a special role in mathematics and particularly in the applications of mathematics. They should realize that the standard algorithms are superb examples and deserve to be studied for that reason alone. They should also realize that being able to construct algorithms that are correct is an essential skill students must have if they are to apply mathematics.*

An oft-overlooked aspect of the algorithms' power is that they enable computation with numbers of unlimited size using only the single-digit "math facts." This also emphasizes to teachers the importance of rapid, automatic recall of those facts—and provides a rare example where rote memorization *is* absolutely necessary.

Note also that the standard division algorithm is an important tool for exploring and understanding rational numbers as repeating decimals, a key step leading to the study of irrational numbers. It is also indispensable for dividing polynomials in algebra.

B.6. Fractions, decimals, percents, and mixed numbers

Understanding, fluency, comfort, and confidence with fractions are critical for progress in elementary mathematics and especially for success in algebra, with its rational expressions, fractional coefficients, fractional exponents, etc. Unfortunately, fractions are also a source of bewilderment and anxiety among students *and* teachers, where the urge to say that $\frac{1}{2}+\frac{2}{3}=\frac{3}{5}$ sometimes appears irresistible.[14]

Much of the trouble stems from the blind-men-and-elephant approach taken by many curricula, which discuss the myriad incarnations and interpretations of fractions without stating what they actually are: *numbers*. They are siblings of the whole numbers, live on the same number line, and share the same properties and operations. CBMS illustrates this common misconception as follows:

> *Many children, and older students as well, see fractions only as pairs of natural numbers plugged into arithmetic procedures. So, for example, in the second National Assessment of Educational Progress, when students were asked to pick an estimate for* 12/13 + 7/8 *from the choices,* 1, 2, 19, *and* 21, *most chose the latter two, presumably having combined either their numerators or their denominators. They failed to recognize that* 12/13 *and* 7/8 *are each quantities close to* 1 *and, thus, their sum is close to* 2.
>
> …
>
> *Most teachers know what a fraction is under at least one of its interpretations, but they often lack a sense of relative size. Having memorized a method for finding a common denominator and comparing numerators, they cannot determine, say, which of a pair is larger—*5/7 *or* 7/9*? or* 5/8 *or* 7/12*?—without applying that procedure. Working with area diagrams, teachers can explore such fraction pairs and learn to use other strategies, e.g., given common numerators, comparing the denominators; or considering how much smaller each fraction is than 1, or how much larger than* 1/2. *Such observations can, in turn, be applied to more cumbersome pairs of fractions.* [2, p. 69]

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[14] Note that this infamous operation is correct for combining batting averages: 1 for 2 yesterday and 2 for 3 today means the batter is now 3 for 5. It's also handy for finding a fraction between two others.

Problems involving ordering, relative size, and the number line are useful for internalizing these points:

Place in order from smallest to largest: 3/8 3/10 3/5 1/4 1/2

Which is larger, 6.51 *or* 6.509999*?* *What is* $2.56789-\frac{1}{2}$*?*

Place on a number line: 0.0333… 1/3 –7/19 1.00001 –0.75 113/102

Find two numbers between $3\frac{3}{4}$ *and* $3\frac{76}{100}$. *How many other numbers lie between these two?*

Teachers and students need to root their thinking about fractions in a single, clear definition. It is strongly recommended [3, 9, T1, T4, T7] that the fraction $\frac{m}{n}$ $(n \neq 0)$ be defined as a *point on the number line*. For example, place new points that divide each interval between consecutive whole numbers into *n* equal parts: the fraction $\frac{1}{n}$ is defined to be the first such point to the right of zero, and the fraction $\frac{m}{n}$ to be the m^{th} such point. Then a useful exercise is to show that other descriptions/interpretations of fractions (quotient, ratio, part-whole, etc.) are equivalent to this definition.

Decimals, percents, and mixed numbers—different ways of writing fractions—are just *more numbers*.

- Decimal fractions illustrate why a complete understanding of *place value* is critical.
- Percents should be easy—substitute $\frac{1}{100}$ for % and you have a number—but they're not (see next paragraph). Many have considerable difficulty with problems like these:

  (a) 75 *is* 30% *of what number? (b)* 30 *is what percent of* 75*? (c) What is* 75% *of* 30*?*
- The implicit + sign in a mixed number is a frequent source of confusion; teachers must understand that $3\frac{1}{2}\times 5\frac{2}{3}$ is hiding an application of the distributive property: $(3+\frac{1}{2})\times(5+\frac{2}{3})$. Conversion to improper fractions is often convenient but *not* as a mandatory procedure to be memorized.

A major obstacle to mastering fractions, percents, and decimals is failure to keep track of what the *whole* is (or the *unit*: see Section B.11). Whenever a fraction refers to one whole and another refers to a different whole, trouble is sure to follow. Problems like these will help drive home the point:

$\frac{2}{3}$ *cup of flour makes* $1\frac{3}{5}$ *batches of brownies. How many batches will* $2\frac{1}{2}$ *cups of flour make?*

How much is $1\frac{1}{2}$ *thirds? What about* $3\frac{1}{7}$ *halves?*

A rancher sold two heifers for $200 each. The first was sold 20% above cost while the second was sold 20% below cost. Did she have a net profit or loss on the two transactions?

Jerry bought two identical pizza pies, each cut into six equal pieces, and ate one piece from each pizza. What fraction did he eat? (This is deliberately ambiguous.)

10%, $\frac{1}{10}$, and 0.1 all represent the same number, but there is a subtle distinction in the symbolism. The % symbol implies (and reminds us) that it must be 10% *of something*. All three forms normally refer to a whole or unit, but in the latter two cases this is easy to overlook. For example, if the rancher/heifer problem above is rewritten in the (deliberately ambiguous) form "The first was sold 0.20 above cost…" one could interpret that as 20 cents rather than 0.2 of the cost.[15]

The operations +, –, ×, ÷ were defined previously for whole numbers; teachers should think carefully about the implications of extending them to this new type of number (see Sections B.4 and B.8). Dividing fractions is notoriously baffling and can be attacked in several ways:

- Dividing on the *number line* builds intuition and helps to break the whole-number habit of assuming that the quotient will be smaller than the dividend.
- Common denominators: $\frac{2}{3} \div \frac{3}{4} = \frac{8}{12} \div \frac{9}{12}$ and 8 twelfths ÷ 9 twelfths is clearly $\frac{8}{9}$.
- Use the fact that $A \div B$ is the solution of $? \times B = A$
- Division is equivalent to multiplying the reciprocal: $A \div B = A \times \frac{1}{B}$, which is the invert-and-multiply rule.

References [3, 6, T1, T3, T4, T7] contain extensive discussions of fractions.

Summary: A good mantra is *"They're all just numbers!"*—accompanied by problems like this:

A car travels 8/9 of a mile in 2/7 *of an hour; how fast is it moving?*
A car travels 60 *miles in* 2 *hours; how fast is it moving?*
Why are so many people stymied by the first question but not the second?

B.7. Signed numbers

The analogous roles of signed numbers and fractions should be made explicit and clearly understood. Each of these "new" types of numbers extends the set of whole numbers, one by filling in the number line to the left of zero and the other by filling in between the whole numbers.

Using the number line, teachers should also think carefully about how the operations extend to these new domains. The arithmetic of signed numbers and the idea that every number A has an "opposite" ^{-}A are fairly intuitive (the analogy to reciprocals and division should be made explicit). Subtracting a negative can be problematic, but even skeptics are usually reassured by the simple principle that

- Subtraction is equivalent to adding the opposite: $A - B = A + {}^{-}B$

The thorniest question that arises in this context is why negative × negative = positive. Teachers will encounter metaphors and manipulatives for justifying signed number operations to students (debt, temperature, hot and cold cubes, walking backward, playing film backward), but those are often unsatisfying where multiplication is concerned. One can also reflect the number line about its origin (see Section B.3) or write a table (
$2 \times {}^{-}3 = {}^{-}6;\ 1 \times {}^{-}3 = {}^{-}3;\ 0 \times {}^{-}3 = 0;\ {}^{-}1 \times {}^{-}3 = 3;$ etc...).

Intuitive arguments notwithstanding, this is a prime opportunity for teachers to begin thinking like mathematicians, by realizing that this odd-looking relationship *must* be true if negative numbers are to satisfy the properties of arithmetic:

Prove that negative × *negative = positive using the distributive property.*
To get started, consider (3–2) × (4–3) *and use the distributive property to find* (–2) × (–3).

[15] Some mathematicians prefer to define percents as ratios, operators, or something else, but this level of abstraction may confuse elementary teachers.

B.8. Number systems (integers, rationals, and reals)

As mentioned above, the discussions of negative numbers and fractions should make explicit the fact that these are *expansions* of the familiar counting numbers—inclusion of zero[16] yields the whole numbers, which form a subset of the integers, which in turn are a subset of the rational numbers.

Any formal development of real numbers is beyond the scope of this course. Nevertheless, a brief peek beyond the rationals is valuable because it completes the nested "Russian-doll" package of number systems and provides another opportunity for teachers to think mathematically. This usually begins by showing that every rational number is equivalent to a repeating (or terminating) decimal and then demonstrating sequences that do not repeat and hence cannot be rational. It is also a challenging but valuable exercise for teachers to show, using the decomposition of integers into prime factors, that $\sqrt{2}$ and numbers like it must be irrational. A further intellectual stretch that is always popular is to demonstrate that there are "infinitely more" real numbers than rationals.

A proof that the operations and their properties extend to real numbers[17] is also out of scope, but it should be made clear that such an extension is not as obvious as it might appear.

B.9. Properties of arithmetic

Elementary curricula typically mention, at various points in time, *some* of the properties of arithmetic:[18]

- The commutative, associative, identity, and inverse properties for addition and multiplication
- The distributive property binding addition and multiplication together

Teachers must understand these properties and recognize when they are applying them. More importantly, they also need to

- See them presented as a *complete, coherent package.*
- Notice the symmetry between additive and multiplicative properties.
- Understand that this simple set of rules governs all of arithmetic.
- Realize that the "inverse" operations of subtraction and division are not on the list because they are *consequences* of the properties.

A solid comprehension of the properties will help teachers evolve a view of elementary arithmetic as a logical, coherent whole, the antithesis of its common perception as a "collection of disconnected facts and procedures to be memorized."

Note: it is important to state explicitly that the distributive property applies to division as well as multiplication, and to provide some exercises that illustrate this point. It is remarkable how many people can handle an expression like $3(9x+3y-27)$ but are tripped up by $\frac{9x+3y-27}{3}$.

[16] Remarkably, zero was not accepted as a legitimate number in Europe until nearly the beginning of the Renaissance.

[17] Wu dubs this the "Fundamental Assumption of School Mathematics." [T4]

[18] Together with closure, these are the *field axioms*, but an axiomatic approach will be too abstract for most students at this level.

B.10. Number sense and mental math

Arithmetic proficiency includes a collection of skills known informally as *number sense* (fluency with calculations, numerical intuition, and the use of estimation to check results) and *mental math* (using mastery of place value and the distributive, commutative, and associative properties to do many computations in one's head.)[19] Teachers need to acquire and gain confidence with these skills—and without recourse to a calculator—so that they can develop them in their own students. For example:

Do each calculation in your head, then explain how you did it:

$12+39+88+61$	48×25	$40{,}000\times2{,}200$	99×17	$72-59$
$1{,}000{,}000\div0.5$	$99+88$	$2\times968\times5$	256×4	$4300\div50$

Which is larger, $\frac{25}{82}$ *or* $\frac{25}{83}$*?* $\frac{99}{100}$ *or* $\frac{100}{101}$*?* $\frac{23}{47}$ *or* $\frac{101}{201}$*?*

What is 15% *of* 80*?* 300% *of* 6*?* 20% *of* 22.222*?* 90% *of* 500*?*

B.11. Units

Units are perhaps the most underappreciated concept in mathematics. An integral part of nearly every calculation, they provide physical context, point the way to problem solutions, and certainly deserve more than the one point they typically receive on a quiz! Equality depends on units (12 inches = 1 foot but $12 \neq 1$) and numbers can be added if and only if they have the same units (2 pigs + 3 pigs = 5 pigs but 2 thirds + 3 fourths is *not* 5 sevenths).[20]

Multiplication and division operate on both numbers and their units[21]:

3 meters $\times$ 2 meters = 6 m^2, a unit of area

60 miles ÷ 3 hours = $20\,\frac{\text{miles}}{\text{hours}}$, a speed

4 kilowatts $\times$ 2 hours = 8 kw-hr, a unit of energy

5 dollars ÷ 2 kilograms = $2.50\,\frac{\$}{\text{kg}}$, a unit cost

This is important because teachers must understand that division problems often create *rates* (miles per hour, $ per pound, bushels per acre), that *proportions* are instances of *constant rates*, and that underpinning all of these are *linear functions* (see next section).

Units also make a terrific problem-solving tool, for example:

How long will a 40 gallon tank take to fill at 5 gallons/minute? Do we multiply or divide? Divide what?

The units show when you're on the right track: to get minutes, divide $\dfrac{40\text{ gal}}{5\ \frac{\text{gal}}{\text{min}}} = \dfrac{40}{5}\ \text{gal} \times \dfrac{\text{min}}{\text{gal}} = 8\text{ min}$

Unit conversions are also easy: just keep multiplying by 1 until the units work out. For example, to convert feet/minute to miles/hour, multiply by $1 = \dfrac{60\text{ min}}{1\text{ hour}}$ and then again by $1 = \dfrac{1\text{ mile}}{5280\text{ feet}}$.

[19] This can even help with such basic tasks as memorizing the times tables: Can't recall 8×6? It's twice 4×6.

[20] This utilizes the Gross brothers' signature metaphor: numbers are *adjectives* that modify nouns and one can add them if and only if they modify the *same* noun. In multiplication, the adjectives and nouns are multiplied separately.[T7]

[21] A good question to open this discussion is "How much is 3 tens times 2 tens?" A popular answer is "6 tens."

Finally, teachers can make good use of the units that lurk inside place value and fractions:

3 hundred $\times$ 2 thousand = 6 hundred-thousand	1 fifth + 2 fifths = 3 fifths
4 hundred $\times$ 7 tenths = 28 hundred-tenths = 28 tens	2 thirds $\times$ 3 fourths = 6 third-fourths = 6 twelfths

B.12. Proportional thinking: rates and ratios

Proportions illustrate how clear definitions can demystify an important concept. Curricula often describe "setting up a proportion" without explaining that a proportion is the tip of an iceberg known as "linear functions." Teachers must understand the relationship among *proportions, constant rates, and linear functions*[22] and be able to explain them to students; this requires fluency with division and units, as well as a working knowledge of basic algebra and functions.

As described above under "Units," division often creates *rates*. For example, if a pen costs $1.50, the rate is said to be 1.50 *dollars per pen* or $1.50\,\frac{\text{dollars}}{\text{pen}}$. If the price is the same *no matter how many pens one buys* (the usual but seldom stated assumption), i.e., if they are sold at a *constant rate* of $1.50\,\frac{\text{dollars}}{\text{pen}}$, then 2 pens cost $3.00, 3 cost $4.50, etc. and the constant rate can be expressed as

$$\frac{1.5\text{ dollars}}{1\text{ pen}} = \frac{3\text{ dollars}}{2\text{ pens}} = \frac{4.5\text{ dollars}}{3\text{ pens}} = \frac{6\text{ dollars}}{4\text{ pens}} = \ldots$$

and any pair of these quotients is commonly called a *proportion*. This can be expressed algebraically as

$$c = 1.5n, \quad \text{where } c \text{ represents the total cost of buying } n \text{ pens}$$

which is (confusingly) sometimes also called a proportion. It is, of course, an instance of a *linear function* $c(n) = 1.5n$, whose graph is a straight line (passing through the origin because there is no constant term) with a *slope* of 1.5 dollars/pen, and each quotient above corresponds to a point on the line. How much of this should be communicated to students depends upon their grade level, but the teacher must understand it well.

Other useful examples and problems involve moving objects, map scales, and similar triangles constructed from flagpoles and shadows, but the usually implicit assumptions (constant speed in a straight line, uniform scales in both directions, $\frac{\text{object height}}{\text{shadow length}}$ = constant if measured at the same time of day) must be explicitly identified.

Along with the more traditional proportion problems, it pays to probe teachers' level of understanding by mixing in a few like these:

> *A* 1,200 *word story averaged five letters per word and had a vowel to consonant ratio of* 3:5. *Approximately how many consonants did this story contain?*
>
> *My dog was* 100 *m from home and my cat was* 80 *m from home. When I called, both ran directly home. If my dog ran twice as fast as my cat, how far from home was my cat when my dog reached home?*
>
> *Sue and Julie were running equally fast around a track. Sue started first. When she had run* 9 *laps, Julie had run* 3 *laps. When Julie had completed* 15 *laps, how many laps had Sue run?*

[T7, Chap. 10] develops these ideas in more detail and [3] includes several discussions of proportions.

[22] Linear functions without constant terms, to be precise.

B.13. Basic number theory

The multiplicative structure of whole numbers (factorization into prime factors) is often poorly understood but teachers must master it nevertheless, and a good measure of their understanding is whether they can find, use, and explain least common multiples (LCM) and greatest common factors (GCF). Familiarity with the divisibility rules is important, but deducing *why they work* is invaluable. The Euclidian algorithm for finding a GCF is another good exercise in mathematical thinking, if time permits. See [3, Secs. 2.8 & 4.15].

As always, questions should be designed to probe understanding, e.g.,

> *Is the product of two perfect squares a perfect square? Explain how you know.*